

# Automorphy of $\mathrm{Symm}^5(\mathrm{GL}(2))$ and base change

Luis Dieulefait\*

Universitat de Barcelona

e-mail: ldieulefait@ub.edu

## Abstract

We prove that for any Hecke eigenform  $f$  of level 1 and arbitrary weight there is a self-dual cuspidal automorphic form  $\pi$  of  $\mathrm{GL}_6(\mathbb{Q})$  corresponding to  $\mathrm{Symm}^5(f)$ , i.e., such that the system of Galois representations attached to  $\pi$  agrees with the 5-th symmetric power of the one attached to  $f$ .

Assuming a slight strengthening of the Automorphic Lifting Theorems that we use in the proof of this result, we note that the same conclusion applies also to any newform without CM of level prime to 30.

We also improve the base change result that we obtained in a previous work: for any newform  $f$ , and any totally real number field  $F$  (no extra assumptions on  $f$  or  $F$ ), we prove the existence of base change relative to the extension  $F/\mathbb{Q}$ .

Finally, we combine the previous results to deduce that base change also holds for  $\mathrm{Symm}^5(f)$ : for any Hecke eigenform  $f$  of level 1 and any totally real number field  $F$ , the automorphic form corresponding to  $\mathrm{Symm}^5(f)$  can be base changed to  $F$  (again, this can be *conditionally* extended to non-CM newforms of level prime to 30).

## 1 Introduction

After completion of our previous work on base change for  $\mathrm{GL}(2)$  (cf. [Di12a]), while explaining the result in a conference at Durham we stressed that “by

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modifying a bit the proof it should follow that every pair of newforms can be connected to each other by a *safe chain*". Moreover, we predicted that such a result should have important consequences, since it should allow to propagate *nice modularity properties* (such as base change) from a single newform to the rest of them.

In this paper, we give a detailed proof of the existence of such safe chains and we give the first applications of this to Langlands functoriality.

The definition of *safe chain* is context-dependent, in all cases it refers to a series of congruences between two different newforms (it is actually better to work with the equivalent notion of congruences between the attached Galois representations), in (*a priori*) arbitrary characteristics, having as initial and final elements the two given newforms. Actually, the congruences are up to twist by finite order characters, and in this paper (reusing an idea already appearing in [Di09]) we also allow replacing a newform by a Galois conjugate of it as a valid *move*.

The important thing about safe chains is that (at each congruence in the chain) they should preserve automorphy of some derived objects: *soit* the restrictions to the Galois group of some number field of the given pair of modular Galois representations, *soit* some symmetric power  $\text{Sym}^n$  of them. By "preserving automorphy" we mean that some available Automorphy Lifting Theorem (A.L.T.) should apply, so as to ensure that IF one of the representations (2-dimensional or  $n + 1$ -dimensional, depending on the case) is automorphic, so is the other. Sometimes (as in [Di12a]) this is needed to work only in one of the two directions (and this can be thought as some sort of inductive process propagating modularity from objects of small "invariants" to those of higher "invariants"), but in this paper we will need this to work in both directions (in order to apply a transitivity argument).

Therefore, the conditions imposed at each link (i.e., congruence) in the chain that makes it *safe* depend on the A.L.T. available: typically we should impose some local condition (specially at the residual characteristic  $p$ ) at both sides of the congruence, and we should ensure that the residual image is sufficiently large.

We allow "congruences up to twist" because the effect of twisting a 2-dimensional Galois representation on its  $n$ -th symmetric power is again a twist, and twists are known to preserve automorphy, and they are harmless because all the technical conditions required in the A.L.T. that we will apply (local conditions, size of residual image) are known to be preserved by twisting by a finite order character. We also allow Galois conjugation because the

$n$ -th symmetric powers of two conjugated Galois representations are themselves conjugated to each other and Galois conjugation obviously preserves automorphy.

The main result in this paper concerns the application of this machinery to deduce automorphy of  $\mathrm{Symm}^5(f)$  for all level 1 cuspforms  $f$ . The result follows from the construction of a suitable *safe chain*, it suffices to prove two things:

- 1) base case: there exists a level 1 cuspform  $f_0$  such that  $\mathrm{Symm}^5(f_0)$  is automorphic.
- 2) safe chain: every pair of level 1 cuspforms  $f$  and  $f'$  can be linked by a chain such that the corresponding chain of 6-dimensional representations linking  $\mathrm{Symm}^5(f)$  with  $\mathrm{Symm}^5(f')$  is a *safe chain* in both directions, in such a way that via suitable A.L.T. we can conclude that  $\mathrm{Symm}^5(f)$  is automorphic if and only if  $\mathrm{Symm}^5(f')$  is automorphic.

Observe that for the 6-dimensional Galois representations to be residually irreducible (this is part of the requirements for the application of all A.L.T.) one should avoid working in characteristics 2, 3 or 5 at all steps.

For part (1), we will consider a cuspform  $f_0$  having, in some large characteristic  $p$ , residual image projectively isomorphic to  $A_5$ , and we will show how to deduce from this residual automorphy of  $\mathrm{Symm}^5(f_0)$ , and then automorphy via a suitable A.L.T.

For part (2), we need to construct a series of congruences linking each given pair of level 1 cuspforms, making sure that at all steps the residual characteristic is greater than 5 and residual images are “large” or even “6-extra large” (even after restriction to  $\mathbb{Q}(\zeta_p)$ ): see the precise definition of these notions at the end of this introduction. Thanks to the results in [GHTT] and in Guralnick’s Appendix A (see [G]) to this paper, this is enough to ensure that the 5-th symmetric powers of these representations (restricted to  $\mathbb{Q}(\zeta_p)$ ) have residually *adequate* image, a condition needed to apply the two main A.L.T. in [BLGGT], theorems 4.2.1 and 2.3.1. In fact, we are relying on a slight strengthening of theorem 4.2.1 of loc. cit. which is proved in Appendix B (written jointly with T. Gee, see [DG]). Thanks to this improvement the

assumption<sup>1</sup>  $\ell \geq 2(n + 1)$  (which is part of condition (4) of this theorem, cf. [BLGGT]) can be replaced by the standard condition of adequacy of the (restriction of) the residual image, as in condition (5) of theorem 2.3.1 in loc. cit.

We also need to make sure that at each step the 5-th symmetric power of both  $p$ -adic Galois representations being considered satisfy (locally at  $p$ ) the local condition needed to apply some A.L.T. in both directions. We will see that in our process they are either both “potentially diagonalizable” or both ordinary, allowing us to apply theorems 4.2.1 (its variant in Appendix B) and 2.3.1 in loc. cit., respectively.

The reader should also notice that these A.L.T. are stated in loc. cit. and in Appendix B for representations of imaginary CM fields, but a standard argument using quadratic base change allows an extension to the case of totally real fields, where representations are now essentially self-dual and automorphy is stated in terms of RAESDC automorphic representations, see for example [CHT].

The fact that our chains are reversible, i.e., that they can be used to propagate automorphy in both directions, is crucial for us, since the way that we proceed to link any pair of level 1 cuspforms is through an inductive process, combined with transitivity. More precisely, we are able to link any pair of level 1 cuspforms  $f$  and  $f'$  by linking both of them to two newforms in the same orbit, i.e., to two conjugated newform  $g$  and  $g^\sigma$ . Since Galois conjugation is a valid move (because it preserves automorphy), this implies that the two given newforms are linked to the same  $g$ , thus by transitivity they are linked to each other: just concatenate the two chains! Thus, if automorphy is known for  $\text{Symm}^5(f')$ , it is propagated to  $\text{Symm}^5(f)$  passing through  $\text{Symm}^5(g)$  somewhere in the chain. It is clear then that A.L.T. should work in both directions, since we need them to propagate automorphy (starting at  $f'$ ) “down” until we reach  $g$ , and then “up” until we reach  $f$ .

Let us stress that the orbit of  $g$  is unique and independent of the given forms  $f$  and  $f'$ : it is a universal step appearing in all safe chains. The inductive argument linking any level 1 cuspform  $f$  with  $g$  will be described in detail in the paper, and it consists of a series of safe congruences that finally lead to a space of newforms of small level and weight, and fixed (up to conjugation) nebentypus, where computations show that there is a unique orbit of new-

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<sup>1</sup>just for this line, we use the notation of [BLGGT]: the residual characteristic is  $\ell$ , and the dimension of the Galois representations is  $n$

forms.

The construction of this safe chain borrows some key ideas from [Di12a] and [Di09], but they have to be modified and generalized in order to be applied in this new context.

Using this safe chain, together with the techniques of “ramification swapping” (as in [Di12b], also used in [Di12a]) and “killing ramification” (as in [Di12b]), we will also be able to obtain an improved proof of base change for  $GL(2)$ . This time the safe chain will connect any cuspform to a CM form, and now “safe” will refer to the fact that Modularity Lifting Theorems (M.L.T.) for the restrictions of the 2-dimensional Galois representations to the Galois group of a totally real number field do apply in both directions at each step of the chain.

We are relying again, for the ramification swapping process, on the two main A.L.T. in [BLGGT] and the improvement in Appendix B, this time for 2-dimensional Galois representations, and the important feature of these theorems that the required local conditions are preserved under arbitrary base change allows us to work over arbitrary totally real fields  $F$ , not needing to impose any local condition. For the “killing ramification” process, we will need to apply the M.L.T. of Kisin in [K], which requires the residual characteristic  $p$  to be split in  $F$ , but since this step will take place at a point where the prime  $p$  to be killed is a suitable auxiliary prime, we just have to check that such primes can be chosen to be split in  $F$  (a similar process took place in [Di12a]).

In order to cover the case of newforms of even level we will also apply a 2-adic M.L.T. of Kisin in [K-2] together with some ideas taken from [KW].

We also prove a base change result for the automorphic forms corresponding to  $\mathrm{Sym}^5(f)$ , which follows easily by combining previous results.

We conclude this introduction by warning the reader that the proof of the main result (the construction of the safe chain) is full of twists and turns, and at several places the path is so full of obstacles that it seems there is no way to continue, and it is only by some completely new trick, or by an unexpected generalization/combination of results in some of our previous works that one manages to keep going. It is a road full of miracles, including a fascinating generalization of Maeda’s conjecture that motivates the choice of the “small single orbit space” of newforms at the bottom of the road. The tricks

of “good-dihedral primes”, “micro good-dihedral primes”, “Sophie Germain primes”, “weight reduction via Galois conjugation”, are basic to build the safe chain.

Both appendices are also of key importance: without them we would have been forced to work only on characteristics  $p > 13$ , and just from the computational point of view (to say nothing of the theoretical complications) it would have been impossible to complete the “low part” of the chain.

Let us now record the two main theorems proved in this paper: in sections 2,3, and 4, and in section 6 for the extension to other fields, we prove:

**Theorem 1.1** *Let  $f$  be a level 1 cuspform. Then if we call  $\rho$  an  $\ell$ -adic Galois representation attached to  $f$ , the 6-dimensional Galois representation  $\text{Sym}^5(\rho)$  is automorphic, namely, there is a cuspidal self-dual automorphic representation  $\pi$  of  $\text{GL}_6(\mathbb{Q})$  (a RAESDC in the notation of [CHT]) such that the  $\ell$ -adic Galois representation attached to  $\pi$  is isomorphic to  $\text{Sym}^5(\rho)$ . Moreover, if we restrict this 6-dimensional Galois representation to any totally real number field  $F$ , it is still automorphic, meaning that it is isomorphic to the  $\ell$ -adic Galois representation attached to a cuspidal self-dual automorphic representation of  $\text{GL}_6(F)$ .*

Furthermore, a conditional extension of this result to non-CM newforms of level prime to 30 is discussed in section 7.

In section 5 we prove base change for classical cuspidal modular forms in full generality:

**Theorem 1.2** *Let  $f$  be a newform of arbitrary level and weight. Let  $F$  be any totally real number field. Then,  $f$  can be lifted to  $F$ , i.e., there is a Hilbert newform  $f'$  over  $F$  whose attached  $\ell$ -adic Galois representations are isomorphic to the restriction to  $G_F$  of those attached to  $f$ .*

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We finish this section with some definitions and notations.

**Notation:** In this paper,  $F$  will always denote a totally real number field. For every number field  $K$ , we will denote by  $G_K$  the absolute Galois group of  $K$ .

We will write  $\zeta_p$  for a primitive  $p$ -th root of unity. We will denote by  $\chi$  the  $p$ -adic or mod  $p$  cyclotomic character. The value of  $p$ , and whether it is the  $p$ -adic or the mod  $p$  character, will always be clear from the context.

We will denote by  $\omega$  a Teichmüller lift of the mod  $p$  cyclotomic character.

Given a Galois representation  $\sigma$ , we will denote by  $\mathbb{P}(\sigma)$  its projectivization.

**Definitions:** Let  $K$  be a number field. Let  $\bar{\rho}_p$  be a two-dimensional, odd, representation of  $G_K$  with values on a finite extension of  $\mathbb{F}_p$ .

1. We say that the image of  $\bar{\rho}_p$  is *large* if  $p \geq 7$  and the image contains  $\mathrm{SL}(2, \mathbb{F}_p)$ , or  $p = 3$  or  $5$  and the image contains  $\mathrm{SL}(2, \mathbb{F}_{p^r})$  for some  $r \geq 2$ . If this is the case, it is easy to see that the image of  $\mathbb{P}(\bar{\rho}_p)$  is isomorphic to one of the following two groups:  $\mathrm{PSL}(2, \mathbb{F}_{p^r})$  or  $\mathrm{PGL}(2, \mathbb{F}_{p^r})$ , for some  $r$  (and  $r \geq 2$  if  $p = 3, 5$ ). This implies in particular that large images are non-solvable, and also that they are adequate subgroups of  $\mathrm{GL}(2, \bar{\mathbb{F}}_p)$ , see theorem 1.5 and remark 1.6 (3) in Appendix A and the main result of [GHTT].
2. We say that the image of  $\bar{\rho}_p$  is *6-extra large* if  $p = 11$  or  $13$  and the image contains  $\mathrm{SL}(2, \mathbb{F}_{p^r})$  for some  $r \geq 2$ .  
For  $p = 7$  or  $p \geq 17$  we say that the image of  $\bar{\rho}_p$  is *6-extra large* if it is large.  
This technical condition is needed to ensure that the 5-th symmetric power of  $\bar{\rho}_p$  has adequate image (see theorem 1.5 and remarks 1.6 (1)).

and 1.6 (3) in Appendix A and the main result of [GHTT]).

**Remark:** Observe that since large and 6-extra large images correspond to almost simple projective images, these conditions are preserved if we restrict the representation to a cyclic extension of  $K$ , such as  $K(\zeta_p)$ .

3. We say that the image of  $\bar{\rho}_p$  is *dihedral* when the image of  $\mathbb{P}(\bar{\rho}_p)$  is a dihedral group of order at least 4.
4. We say that the image of  $\bar{\rho}_p$  is *bad-dihedral* when it is dihedral,  $p > 2$ , and the restriction of  $\bar{\rho}_p$  to the absolute Galois group of  $K(\sqrt{\pm p})$  becomes reducible, where the sign is  $(-1)^{(p-1)/2}$ .

## 2 Construction of the safe chain: a general description

This is the *code description* of what is needed to make the safe chain. Recall that this is just “half” of the chain, the full chain is obtained just by concatenation of two of these safe chains, which makes sense because both end up in the same orbit, and are reversible.

At each step we describe the nature of the movements to be performed, and the output. References are to indicate where some of these tricks first appeared.

We start with a cuspform  $f$  of level 1 and weight  $k \geq 12$ :

- 1) Introduce Good-Dihedral prime [KW]: level raises to  $q^2$ .
- 2) Weight reduction via Galois conjugation [Di09]: weight reduces to  $2 < k \leq 14$  ( $k$  even).
- 3) *Ad hoc* tricks to make the small weight congruent to 2 mod 3 (Sophie Germain primes, Hida families...): end up with  $k \equiv 2 \pmod{3}$  and  $k < 43$ .
- 4) Introduce MGD prime 43 using the pivot primes 7 and 11 [Di12a]: end up with newform of weight 2 and level  $43^2 \cdot q^2$ .
- 5) Remove the Good-Dihedral prime (in two moves): end up with a newform of weight  $q + 1$  and level  $43^2$ .
- 6) Again weight reduction via Galois conjugation but this time “highly improved”, because we need to ensure large residual image at each step using



just the MGD prime. Weight reduces to  $2 < k \leq 14$  ( $k$  even).

7) *Ad hoc* tricks to make weight smaller than 17 and divisible by 4 (Sophie Germain primes, Khare’s weight reduction, non-dihedrality due to class field theory of real quadratic field...): end up with a newform of weight 16 and level  $43^2$ .

8) Introduce nebentypus at 17 of order 8: end up with a newform of weight 2, level  $43^2 \cdot 17$ , with nebentypus.

9) Remove the MGD prime modulo 11 via an *ad hoc* Lemma to ensure residual irreducibility: get congruence (maybe using level-raising) with a newform Steinberg at 43, of weight 2 and level  $43 \cdot 17$ , with nebentypus of order 8 at 17.

10) Move from weight 2 to weight 44 by reducing modulo 43: irreducibility checked by hand.

11) Generalized Maeda in weight 44, level 17, nebentypus of order 8: check that this space has a unique orbit.

Several steps use tricks appearing in the referred paper, and they will be easy to follow for those who have read these papers. Steps (3) and (7) are painful and technical: the weight has been made small, BUT we require some technical condition on it to be able to perform the next step (i.e., steps (4) and (8)): step (3) is done because at step (4) we want  $k \equiv 2 \pmod{3}$  because the prime 43 is not truly Sophie Germain, there is a 3 dividing  $43 - 1$ , thus when we take a modular weight 2 lift in characteristic 43 IF WE KNOW that  $k \equiv 2 \pmod{3}$ , the nebentypus, which is  $\omega^{k-2}$ , will be a character of order 7, and this allows us to work with 43 “as if it were a Sophie Germain prime”, and using this character of order 7, and later the factor  $11 \mid 43 + 1$ , we manage to introduce 43 as a MGD prime (compare with the introduction of 7 as a MGD prime in [Di12a]). Step (7) is done because before moving to characteristic 17 in step (8) with a weight  $k$  smaller than 17 to take a modular weight 2 lift, since we want the character  $\omega^{k-2}$  to be of order 8, we NEED to ENSURE that  $k$  is divisible by 4.

Both in steps (3) and (7), in order to “reduce” to weights satisfying the required conditions, we make a few technical moves that allow “magically” to control the weight and force it to satisfy them.

Step (6) is perhaps the best technical innovation in the paper: we need to perform the weight reduction to inductively go from an arbitrarily large weight  $k$  to  $k \leq 14$ , we do this using the method of [Di09] as in step (2), but this time the level contains just the MGD prime 43, and still we need to ensure

that in the whole process of weight reduction all residual images are going to be large! We succeed thanks to an extra degree of freedom, previously unexploited, in Galois conjugations: there is not a unique conjugation that reduces the weight, there are at least two completely different choices (this we prove), and we can show that for at least one of them the residual representation will have large image.

Step (9), if it were performed by direct computation, will not be surprising. But since the level at this step is  $17 \cdot 43^2$  and the nebentypus of order 8, computations seem out of reach. We prove irreducibility modulo 11 (recall that ramification at 43 being of order 11, at this step one is losing the MGD prime from the level) by using in a non-trivial way information on the ramification at 17 and on the trace at 43 of the residual representation, assumed reducible, to derive a contradiction.

The final step (11) is a direct computation to check that certain space has a unique orbit, but this does not come out of the blue: we “knew” that there should be a unique orbit there by a precise generalization to arbitrary level of Maeda’s conjecture.

### 3 The eleven steps of the safe chain

Before describing each step of the chain, let us make some general remarks. First of all, let us stress that in all steps we are always working in residual characteristic  $p \geq 7$ . This is necessary because we want the 5-th symmetric power to be residually irreducible.

At each step, we will introduce several congruences between modular Galois representations. Since we want to apply the two main A.L.T. in [BLGGT], Theorems 4.2.1 (its variant in Appendix B) and 2.3.1, in both directions, to the 5-th symmetric powers of these representations, at each of the congruences introduced we need to ensure that both 6-dimensional representations are “potentially diagonalizable” (PO-DI) or that they are both ordinary. Observe that both of these local properties are preserved by taking symmetric powers, so we just have to check them at the level of the modular 2-dimensional Galois representations (for the PO-DI property, just observe that if two points in a local deformation ring  $R_{\mathcal{D},p}$  are in the same irreducible component, when we switch to a larger deformation ring  $R_{\mathcal{D}',p}$ , i.e., with the latter ring projecting into the former one, the corresponding points in

the larger ring are also in the same irreducible component: this property is already mentioned in [BLGGT] to show that this notion is preserved when restricting the Galois representations). This will also be useful when we address, in latter sections, the problem of base change, since for that case we are going to rely on the same A.L.T. most of the time, but this time applied to 2-dimensional Galois representations.

Most of the congruences that we will perform involve taking weight 2 modular lifts, for a residual representation in some odd characteristic  $p$  having Serre's weight  $2 < k < p + 1$ , and the congruence will be between a potentially Barsotti-Tate representation and a crystalline representation in the Fontaine-Laffaille range (i.e., of weight  $k \leq p$ ). Since in both cases (by the results of Kisin in [K-BT] in the first case, as proved in [GK], and by [BLGGT] and [GL] in the other) such representations are known to be PO-DI, the local conditions needed to apply Theorem 4.2.1 in [BLGGT] and its strengthening in Appendix B are satisfied.

The second (and last) case that we will encounter in our chain is when one of the representations is semistable at  $p$  of weight 2, and the other one is crystalline at  $p$  of weight  $p + 1$ . In this case, both representations are known to be ordinary (in fact, they live in the same Hida family), thus again we can apply an A.L.T. in [BLGGT], this time Theorem 2.3.1, the one for ordinary representations. At one step (step 3) there is a small variant of this: we consider a congruence between a weight 2 semistable representation and a crystalline representation whose weight is actually larger than  $p + 1$ : we take a suitable specialization of the Hida family at a larger weight  $k \equiv 2 \pmod{p - 1}$ . Again, this is known to be an ordinary representation, so this A.L.T. applies.

The reader can check (this is automatic) that in the 11 steps that follow we will always be in one of the two cases above, thus from now on we are not going to insist anymore on this point: the local conditions required to apply the A.L.T. to the 5-th symmetric powers of our representations are satisfied. Also, when in latter sections we discuss base change, since these local properties are preserved by base change, both for the 2-dimensional representations and for their symmetric powers, the local conditions to apply the A.L.T. over any totally real number field  $F$  are satisfied.

The other obstacle to apply the A.L.T. in [BLGGT] (recall once again that concerning theorem 4.2.1 in loc. cit., we are relying on the improvement that we prove on Appendix B, cf. [DG]) is the condition that residual images should always be adequate. We will check that in our chain residual images

will always be 6-extra large (this is not so at the “base case”, but in this case the image is known to be adequate, too), a condition that is preserved after restriction to the Galois group of  $\mathbb{Q}(\zeta_p)$ , and this implies that the images of these restrictions are adequate, thanks in particular to the results in Appendix A.

The required condition on the residual images will be obtained, at most steps, by using some ramification information. This will be particularly easy when a Good-Dihedral prime is in the level, and at the very bottom of the chain, when no significant ramification is preserved, will be checked partly by hand. In any case, let us stress that in what follows we are just required to indicate two things at each step:

- i) how the chain is built, and
- ii) how it is checked that the residual images are 6-extra large

Observe that since 6-extra large implies large, when we re-use this chain for base chain, we at least know that it is built with large residual images (and we will just have to check that this largeness is preserved by base change).

### 3.1 Step 1

We start with  $f$  of level 1 and weight  $k \geq 12$ . At this step we introduce a Good-Dihedral prime. This is done just as in [Di12a] (and also in [KW] where the idea first appeared), so we will be very brief. We first choose a prime  $r > 13$  larger than  $k$  where the residual image is large, thus 6-extra large, and reducing modulo  $r$  we switch to a weight 2 situation. Here we are introducing a nebentypus at  $r$ , to be removed later. Then, we take primes  $t$  and  $q$  larger than  $r$  as in Lemma 3.3 of loc. cit., and as in the discussion thereafter: by working modulo  $t$ , where the image is  $\mathrm{GL}(2, \mathbb{F}_t)$ , thus 6-extra large, we produce a congruence with a newform  $f_2$  of weight 2 with  $q^2$  in the level, such that ramification at  $q$  is given by a character of order  $t$ , and  $t$  divides  $q + 1$ . The prime  $q$  is a supercuspidal prime for  $f_2$ . Before choosing  $t$  and  $q$ , a large bound  $B$  is fixed, greater than  $k$ ,  $2 \cdot r$ , and than any other auxiliary prime that one will use (such as  $p = 11, 43$ ) and  $t$  and  $q$  are chosen to be larger than  $B$ . We take  $B > 68$  because we are going to need this in Step 3.

These two primes are also asked to satisfy:

$t \equiv 1 \pmod{4}$ ,  $q \equiv 1 \pmod{8}$ , and also, for every prime  $p \leq B$ , we require that  $q \equiv 1 \pmod{p}$ .

In the following steps, up to step 4 (included), we are going to work all the time in characteristic  $p \leq B$ , and this implies that the local ramification at  $q$  will be preserved through these steps. This, together with the above conditions on  $q$  implies, and this is the main point about Good-Dihedral primes (cf. [KW], [Die12]), that the residual images will be large. Moreover, since we are taking  $t > B > 2 \cdot r > 13$  and the residual projective images contain an element of order  $t$  (given by the image of the inertia group at  $q$ ) it is obvious that if we are in characteristic 11 or 13 (and trivially in any other characteristic  $7 \leq p \leq B$ ) the residual image will also be 6-extra large.

We conclude the first step by moving back to characteristic  $r$ . We take a mod  $r$  Galois representation attached to  $f_2$  and we lift it to a modular representation attached to a newform  $f_3$  without  $r$  in the level. As explained in the introduction, we freely twist when necessary, so we are assuming that the Serre's weight of this residual representation is at most  $r - 1$  (and greater than 2 because the nebentypus of  $f_2$  at  $r$  was not trivial), and  $f_3$  is taken to be of this weight. Therefore, we end up with  $f_3$  of level  $q^2$ , Good-Dihedral at  $q$ , and weight  $2 < k < B/2$ .

## 3.2 Step 2

At this step we perform the Weight Reduction via Galois Conjugation (WRGC), exactly as in [Di09] (except that when the weight is 10, we are not reducing it because we don't need to), so as to reduce to the case  $k \leq 14$ .

As explained in the previous step, since we are going to work in characteristics  $p < B$  and we have a large Good-Dihedral prime in the level, residual images are all going to be 6-extra large.

Observe that since  $k < B/2$ , we are allowed to work with primes larger than  $k$  as long as they satisfy  $p < 2 \cdot k < B$ , and this is the case in the process of WRGC.

We also note for the reader convenience that, together with Galois Conjugations (an allowed move, see the introduction), this weight reduction involves congruences between potentially Barsotti-Tate and Fontaine-Laffaille type Galois representations, exclusively.

Since the process is described in [Di09], we are going to be very brief:

Suppose  $k > 14$  (if  $2 < k \leq 14$  we can go directly to step 3). It is perhaps worth recalling that through all this process weights are going to be even, and always larger than 2.

Let  $p$  be the smallest prime larger than  $k$ , except in the case  $k = 32$  where we take  $p = 43$ .

Consider the mod  $p$  Galois representation attached to  $f_3$ , and take a modular weight 2 lift of it, corresponding to a newform  $f_4$  having nebentypus at  $p$  given by  $\omega^{k-2}$ , a character of order  $m = (p-1)/d$ , where  $d = (p-1, k-2)$ . Let us call  $a = (k-2)/d$ .

The level at  $p$  of  $f_4$  is  $p$  and it is principal series at  $p$ , with ramification of the attached  $\ell$ -adic Galois representations at  $p$  (and type of the  $p$ -adic Galois representation) being given by  $\omega^{k-2} \oplus 1$ .

Let us assume (this will be proved later) that  $m > 6$ , and let us consider  $t$  to be the smallest integer greater than  $m/2$  and relatively prime to  $m$ . Observe that the difference  $t - m/2$  is at most 2, and that  $t < m - 1$ .

The Galois conjugation is meant to change the nebentypus: we take an element  $\gamma$  in the Galois group of  $m$ -th root of unity corresponding to raising to the  $a^{-1} \cdot t$ , where the inverse of  $a$  is taken modulo  $p-1$ . This changes the nebentypus (and the local type at  $p$ ) from  $\omega^{k-2} = (\omega^d)^a$  to  $(\omega^d)^t$ .

Now we consider the mod  $p$  Galois representation attached to the conjugated newform  $f_4^\gamma$ . Its Serre's weight can be shown to be (after suitable twisting) either  $k_1 = dt + 2$ , or  $k_2 = p + 1 - dt$ , two values whose sum is  $p + 3$ .

The idea is that because of a strong version of Bertrand's postulate  $p$  will be "near"  $k$ , and this will imply that  $k_1$  and  $k_2$  are smaller than  $k$ : in fact, it is easy to see that  $k_2 < k_1$ , so it is enough to check this for  $k_1$ , and since  $k_1$  is approximately  $d \cdot t$ , and this is approximately  $(d \cdot m)/2$ , and this is approximately  $p/2$ , it is clear than having  $p$  near  $k$  the new weight will be smaller than the given  $k$ .

More precisely: for  $k \geq 38$ , it is known that if we consider the primes nearest  $k$ :  $p_n < k < p_{n+1}$ , it holds:

$$p_{n+1}/p_n < 1.144 \quad (*)$$

And also its obvious consequence:

$$(p_{n+1} - 1)/(p_n - 1) < 1.15$$

From this, calling again  $p = p_{n+1}$  the first prime after  $k$ , we have:

$$(p-1)/(k-2) < 6/5$$

And this implies that the order  $m$  of the nebentypus  $\omega^{k-2}$  is greater than 6 (this we had assumed before, now we are proving it for  $k > 36$ ).

Having this, we define  $t$  as above and Galois conjugate using it as above and easily deduce that for the any of the two possible Serre's weights  $k' = k_1$  or  $k_2$  of the Galois conjugated residual representation it holds:  $p/k' > 1.144$ .

In fact, since  $k_2 < k_1$  it is enough to check this for  $k' = k_1$ , and then it reads:  $p/k' = p/(dt + 2) = (md + 1)/(dt + 2)$ . That this is larger than 1.144 can be easily checked once the precise value of  $t$  is introduced: there are three cases, depending on whether  $m$  is 0 or 2 modulo 4, or odd, but in any case the above quotients give increasing functions (increasing with  $m$ ) that already satisfy the inequality for the smallest values of  $m$  (8, 10 and 7, respectively) and for every  $d$ , thus we conclude that the inequality holds in general.

Because of (\*), this implies:  $k' < p_n$ , thus in particular  $k' < k$ . Thus the induction works.

If  $14 < k \leq 36$ , it can be checked by hand that the above process also works, exactly as described, and always gives a smaller Serre's weight  $k'$  (and it is easy to see that it always gives even values, and that they are always greater than 2).

The only subtle point is that for  $k = 32$  one should take  $p = 43$ .

Let us include here, for the reader's convenience, four examples, including the one of weight 32:

- $k = 16, p = 17$ :  $d = 2, m = 8, t = 5, dt = 10$ ; thus:  $k' = 12$  or 8.
- $k = 18, p = 19$ :  $d = 2, m = 9, t = 5, dt = 10$ ; thus:  $k' = 12$  or 10.
- $k = 30, p = 31$ :  $d = 2, m = 15, t = 8, dt = 16$ ; thus  $k' = 18$  or 16.
- $k = 32, p = 43$ :  $d = 6, m = 7, t = 4, dt = 24$ ; thus  $k' = 26$  or 20.

This concludes the weight reduction, we end up this step with a newform  $f_5$  of level  $q^2$ , Good-Dihedral at  $q$ , of weight  $2 < k \leq 14$ .

### 3.3 Step 3

As it will be clear in Step 4, before introducing the Micro-Good-Dihedral (MGD) prime 43 in the level, we need to reduce to a situation in which the weight not only is small (smaller than 43 will be enough) but also is congruent to 2 modulo 3. Here we combine some ad hoc tricks that allow us to do

so.

The ideas in this step are the following: (in logical order, which, as usual, does not coincide with the final ordering on paper):

If you happen to have some weight  $k$  satisfying  $k \equiv 2 \pmod{3}$  and you believe that there is a prime  $p > k$  “near”  $k$  such that  $p \equiv 1 \pmod{3}$ , then you can check (it works in the few applications that we are going to do now, because it always works!) that if you do weight reduction using  $p$ , as in Step 2, the new (smaller) values of the Serre’s weight,  $k_1$  and  $k_2$ , will both be again congruent to 2 modulo 3 (it makes sense because they sum up  $p + 3$  and this is congruent to 1  $\pmod{3}$ ).

Thus, the strategy is the following: we use the Sophie Germain trick, taken from [Di12b], this allows you to, given a pair  $p_1, p_2 = 2p_1 + 1$  of Sophie Germain primes, reduce all cases of even weights  $2 < k < p_2$  to the **unique** case of weight  $p_2 + 1$ . Unfortunately, because we are forced to take  $p_1 \neq 3$  (we are not allowed to work on characteristics  $\leq 5$ ), this is not satisfactory because  $p_2 + 1$  will never be congruent to 2 modulo 3, but moving to a weight higher than  $p_2 + 1$  (using Hida families) we will remedy this problem. After this step, we will end up with a unique weight, it will be congruent to 2 modulo 3, and we will complete the step by reducing it, as indicated above, in such a way that the mod 3 property on the weight is preserved.

We are going to work all the time in characteristics  $p \leq 67$  (and, as usual,  $p > 5$ ) and with weights  $k \leq 68$ , thus since  $68 < B < t < q$  the Good-Dihedral prime in the level will ensure that residual images are 6-extra large at each step.

The Sophie Germain pair that we consider is given by  $p_1 = 11$ ,  $p_2 = 23$ . As in [Di12b], since the weight of  $f_5$  satisfies  $2 < k \leq 14$ , we move to characteristic 23, reduce mod 23, and take a modular weight 2 lift, having a nebentypus  $\omega^{k-2}$  of order 11. Then we move to characteristic 11, reduce mod 11, and take another modular weight 2 lift, where we have changed the type at 23. Since ramification at 23 was given by a character of order 11, we know that the residual representation will either be unramified at 23 or will have semistable (i.e., unipotent) ramification at 23. Moreover, in the first case it can be checked (a similar situation can be found in [Di12a]) that Steinberg level-raising at 23 can be performed. Thus we conclude that in any case we can take a modular weight 2 lift of this mod 11 representation that it Steinberg at 23.

In the usual Sophie Germain trick, at this step, reducing modulo 23 one manages to create a congruence with another newform of weight 24. But since



such a weight is not useful for our purposes, we take another specialization of the Hida family containing this semistable weight 2 representation: we take  $k = 68$ . Since  $68 \equiv 2 \pmod{22}$  we know that there is an ordinary Galois representation attached to a newform of this weight and level  $q^2$  which is congruent modulo 23 with the given weight 2 representation.

At this point, we have reduced the case of weight up to 14 (greater than 2) to the case of weight 68. Since  $68 \equiv 2 \pmod{3}$ , let us try our weight reduction. We must work only with primes that are congruent to 1 modulo 3. Our first try is  $p = 73$ , using it we manage to reduce the weight but we get stuck in the next step (with  $k' = 44$ : no primes congruent to 1 mod 3 show up “nearby”!), so let us try with  $p = 79$  instead.

We reduce modulo 79, and we take a modular weight 2 lift, with nebentypus  $\omega^{66} = (\omega^6)^{11}$ , of order  $m = 13$ .

Galois conjugation is done as in Step 2, using  $t = (m + 1)/2 = 7$ , i.e., exponentiating to the  $11^{-1} \cdot 7$  in the cyclotomic field of 13-th roots of unity, but this leads us to a value of  $k_1 = dt + 2 = 6 \cdot 7 + 2 = 44$  where, again, we get stuck.

Luckily, this is not the only possible Galois conjugation (this extra degree of freedom will be exploited more systematically at Step 6), so let us try with other value of  $t$ , always prime to 13 of course. Taking  $t = 8$  the values we obtain are  $k_1 = 50$  and  $k_2 = 32$ . Observe that both values are again congruent to 2 modulo 3 (as we explained, it is easy to see a priori that this will hold, due to our careful choice of the prime  $p$ ).

Since the first of these two values is too big (our goal includes the condition  $k < 43$ ), assuming we end up with  $k = 50$  we iterate the process. We take  $p = 61$ , reduce mod 61, and this time the nebentypus of the modular weight 2 lift is  $\omega^{48} = (\omega^{12})^4$ , whose order is 5.

Remark: in Step 2 we always had  $m > 6$ , but  $m = 5$  is also fine for the Galois conjugation trick because it also satisfies  $\phi(m) > 2$  (compare with [Di09], where in the case of  $k = 10$  weight reduction was done with nebentypus of order 5. On the other hand, it is easy to see that if  $m = 6$  or  $m < 5$  since  $\phi(m) \leq 2$  the method can not work).

Taking  $t = 3$  for the Galois conjugation, we easily compute  $k_1 = 12 \cdot 3 + 2 = 38$  and  $k_2 = 26$ .

The conclusion is that we have linked the given newform  $f_5$  of small weight, to another newform with the same level  $q^2$ , Good-Dihedral at  $q$ , whose weight is equal to 26, 32 or 38.

The three values are congruent to 2 modulo 3, and are smaller than 43.

### 3.4 Step 4

Thanks to the tricks in the previous step, we can incorporate 43 as a MGD prime, exactly as we did in [Di12a] to incorporate the MGD prime 7. Recall (cf. [Di12a]) that given a prime  $p$  that is the large one in a Sophie Germain pair, if we call  $p_0 = (p - 1)/2$ , assumed prime, and we take another prime  $p_1$  dividing  $p + 1$ , starting with an even weight  $2 < k < p$ , we had devised a procedure to introduce  $p$  as a MGD prime in the level, in other words, to link the given modular representation with another one, this time having  $p$  as a supercuspidal prime in the level, with ramification given by a character of order  $p_1$ , and having weight 2.

This is explained in detail in [Di12a], it exploits different known cases of congruences between principal series, Steinberg, and supercuspidal modular forms. First via a modular weight 2 lift in characteristic  $p$  we introduce nebentypus at  $p$  of order  $p_0$  (as we did in the previous step with the pair 11 and 23), then in characteristic  $p_0$  we transform this into Steinberg ramification at  $p$  (again, as in the previous step), and finally, moving to characteristic  $p_1$  we can create a non-minimal lift that transform the type at  $p$  into supercuspidal with ramification of order  $p_1$ . As with Good-Dihedral primes, recall that this is useful because for a compatible system containing such a local parameter, the residual representations in characteristics  $\ell \neq p, p_1$  are going to be irreducible (because they are so locally at  $p$ ).

This is almost exactly what we do to introduce the MGD prime  $p = 43$ , using the auxiliary primes  $p_0 = 7$  and  $p_1 = 11$ , to end up with supercuspidal ramification of order 11 at 43. The only difference is that 43 is not Sophie Germain, i.e.,  $(43 - 1)/2 = 21$  is not a prime. But this is precisely why we have made Step 3. The Sophie Germain trick, devised in the first place to introduce Steinberg ramification at  $p$ , is based on the fact that, after making a modular weight 2 lift in characteristic  $p$  the nebentypus  $\omega^{k-2}$  has order  $p_0$ , a prime if we are in a true Sophie Germain situation (observe that we have, as will always be the case in this paper, even weights all through the weight reduction process). Since we have forced the weight  $k$  to be congruent to 2 modulo 3 (this is what Step 3 is about), we can work “as if 43 was Sophie Germain”, the only important think is that the nebentypus  $\omega^{k-2}$  must have prime order, and this is satisfied because since  $k - 2$  is divisible by 6, this nebentypus has order exactly 7 (and, luckily, 7 is a prime!), thus all the pro-

cess of introducing the MGD prime can be done exactly as in [Di12a] thanks to the extra information on the weight.

We stress that all auxiliary primes must be taken to be greater than 5: this is the reason why we have not been able to find a reasonable small true Sophie Germain prime good for our purposes.

As in previous sections, the Good-Dihedral prime  $q$  with  $q > t > B > 68 > 43$  is enough to guarantee that all residual images in the above process are 6-extra large.

After these manipulations, we end up with a newform  $f_6$  having weight 2 and level  $43^2 \cdot q^2$ , Good-Dihedral at  $q$ , and having 43 as a MGD prime.

### 3.5 Step 5

At this step we are going to remove the Good-Dihedral prime  $q$  from the level. In order to do so, we are first going to move to characteristic  $t$ , and then to characteristic  $q$ . Precisely because we are leaving the safety zone of characteristics up to  $B$  we must be careful because in both cases, and in all the steps of the chain that follows, we need to check that the residual images are 6-extra large: we don't have any longer the Good-Dihedral prime to ensure this.

Moreover, when reducing modulo  $t$  we are losing the supercuspidal ramification at  $q$ , so we need other arguments even to justify that this residual representation, and those in the next steps of the chain, are irreducible.

On the other hand, we have also introduced the MGD prime 43 in the level, and as we already mentioned, as long as we avoid working in characteristics 43 and 11, the local information at 43 is enough to guarantee that residual images are (absolutely) irreducible. As the reader can easily check, in this step, and also in Steps 6,7 and 8, we are going to avoid these two characteristics, so residual irreducibility is guaranteed by the MGD prime in these four steps.

Also, ramification at 43 has order 11, and it gives an element of order 11 also in the image of the projective representation, thus the exceptional cases in Dickson's classification of maximal subgroups of  $\mathrm{PSL}(2, \mathbb{F})$ , where  $F$  is a finite field of characteristic  $p$ , is ruled out for the images of the residual representations appearing in the chain, as long as the MGD prime stays in the level (again, this will be the case in Steps 5,6,7 and 8).

Therefore, in this step and the next three steps, because of Dickson's classification and thanks to the MGD prime, in order to ensure that a residual

image is large all that we will have to check is that it is not dihedral.

Moreover, since we are NOT going to work in characteristic 11 during these four steps, and using again the fact that we have an element of order 11 in the projective image, we know that if the residual image is large, it will also be 6-extra large (this is a non-empty statement only for  $p = 13$ ).

This being said, let us proceed to remove the Good-Dihedral prime from the level, checking that residual images are not dihedral.

We start with the newform  $f_6$ , we move to characteristic  $t$  and we reduce modulo  $t$ . Suppose that the residual representation is dihedral, and let  $K$  be the quadratic number field such that the restriction to  $G_K$  becomes reducible. The only ramified primes for the residual representation are  $t, q$  and 43 (at most). Having reduced modulo  $t$ , the ramification at  $q$  will be either trivial or semistable (unipotent). In either case, clearly  $K$  can not ramify at  $q$  (moreover, from the assumption of dihedral image we can conclude that the residual representation is unramified at  $q$ ). On the other hand, the order of ramification at 43 is odd, thus clearly  $K$  can not ramify at 43. We conclude that  $K$  is the quadratic number field ramified only at  $t$ , in other words, that the residual representation is bad-dihedral.

But since the newform  $f_6$  has weight 2 and  $t$  is not in its level, the Serre's weight of the residual representation is  $k = 2$ , and then the residual representation can not be bad-dihedral because  $t > 3$  (cf. [Ri97]). The result of Ribet that we are applying here, a result that is part of the proof of “generically large images for modular non-CM Galois representations” (cf. [Ri85]), and is explained in detail in [Ri97] for the case of weight 2, but holds in general with the same proof, is going to be required several times in the following sections, so let us record it here:

**Lemma 3.1** *Let  $\bar{\rho}$  be an irreducible representation in characteristic  $p$ , for a prime  $p > 2$ , that is attached to some newform  $f$ . Suppose that the Serre's weight  $k$  of  $\bar{\rho}$  satisfies  $2 \leq k \leq p+1$ . Suppose that  $\bar{\rho}$  is bad-dihedral, or, more generally, that it is dihedral and induced from a quadratic number field that ramifies at  $p$ .*

*Then it must hold:  $p = 2k - 1$  or  $p = 2k - 3$ . Moreover, in the first case the representations is reducible locally at  $p$ , while in the second case it is, locally at  $p$ , induced from a ramified character of the unramified quadratic extension  $\mathbb{Q}_{p^2}$  of  $\mathbb{Q}_p$ .*

This result of Ribet, that has been used many times in the literature (for example in [DM], or in [Kh]), can be easily proved by observing that in the case under consideration the projective image of the inertia group at  $p$  must have order 2. The result follows easily from this and the definition of Serre's weight. The assumption on the size of  $k$  is not serious, because it is known that by suitable twisting it is always satisfied, and we always do apply this twisting to reduce to this situation, in all steps of the proof.

We have shown that the mod  $t$  reduction of  $f_6$  has 6-extra large image. On the other hand, we know that it is either unramified or semistable locally at  $q$ . Moreover, in the first case, due to the existence of the lift given by  $f_6$  which is supercuspidal at  $q$  it is easy to see that the well-known necessary condition for Steinberg level-raising is satisfied. Thus, in both cases, doing level-raising if necessary, we consider a lift of this mod  $t$  representation corresponding to a newform  $f_7$  of weight 2, level  $43^2 \cdot q$ , Steinberg at  $q$ .

Now we move to characteristic  $q$ , and we reduce modulo  $q$ . Using once again the MGD prime 43 in the level, we know that the residual image will be 6-extra large provided that it is not dihedral. Again, if it were dihedral, it can not be induced from a quadratic field  $K$  ramified at 43 because the order of ramification at this prime is odd, thus it should be bad-dihedral. But since  $q$  is a Steinberg prime in the level, the Serre's weight  $k$  of this mod  $q$  representation is either  $q + 1$  or 2, and this in any case by applying Lemma 3.1 we know contradicts the fact that the representation is bad-dihedral. We thus conclude that the residual representation can not be dihedral, and thus that it has 6-extra large image.

To conclude this step, we consider a lift of weight  $q + 1$  of this residual representation, given by the specialization to weight  $q + 1$  of the Hida family containing the  $q$ -adic representation attached to  $f_7$ . This lift corresponds to a newform  $f_8$  of level  $43^2$  and weight  $q + 1$ , thus locally at  $q$  it is ordinary.

### 3.6 Step 6

At this step we need again to perform the WRGC as in Step 2, in order to reduce the (very large) weight  $k = q + 1$  of  $f_8$  to a weight  $k \leq 14$ . In all this process, we are going to avoid characteristics 43 and 11, so that

to ensure that the MGD prime 43 is preserved. But we need to ensure that images are 6-extra large in the whole process. In Step 2 this was ensured by the Good-Dihedral prime in the usual way (cf. [KW], [Di12b], [Di12a]), but now we only have the MGD prime in the level.

As explained in the previous section, thanks to the MGD prime, all that we need to check is that residual images are not dihedral. Moreover, in all the process of WRGC, we will work with mod  $p$  Galois representations ramified only at  $p$  and at 43, and since the order of ramification at 43 is odd, the only possible dihedral case is the bad-dihedral case.

Thus, we have to perform WRGC, perhaps with some modifications with respect to the classical version given in Step 2, in such a way that we can guarantee that at each congruence the residual image is not bad-dihedral.

The process of weight reduction will follow a similar pattern as in Step 2: we have a newform  $f$  of weight  $k$  and level  $43^2$ , such that 43 is a MGD prime, and assuming that  $k > 16$ , we take  $p$  to be the first prime larger than  $k$ , except for  $k = 42$  where, since 43 is forbidden, we are going to specify later what prime to take, and also for  $k = 32$  where in Step 2 we choose  $p = 43$  and here, again, we are going to have to choose another prime.

We reduce modulo  $p$ , and since it is easy to check (by hand for small values of  $k$ , and because of the strong version of Bertrand used in Step 2 for  $k > 36$ ) that we always have  $p \neq 2k - 1, 2k - 3$  we apply Lemma 3.1 and conclude that this mod  $p$  representation is not bad-dihedral.

The problem is after doing the Galois conjugation: we can try formally to proceed exactly as in Step 2. With the same notation and same reasonings, we can take a modular weight 2 lift of the mod  $p$  Galois representation, we know that the order  $m$  of the nebentypus  $\omega^{k-2}$  is greater than 6, and we can take  $t$  as before, the first number greater than  $m/2$  relatively prime to  $m$ , and use it to Galois conjugate. The problem is that after doing so, the new Serre's weight of the mod  $p$  reduction will be (after suitable twisting) equal to  $k_1 = dt + 2$  or to  $k_2 = p + 1 - dt$  (both smaller than the given  $k$ , as shown in Step 2), and in some cases it WILL BE THE CASE that some of these two values, call it  $k'$ , will satisfy  $p = 2k' - 1$  or  $p = 2k' - 3$ , thus we can not apply Ribet's Lemma to rule out the bad-dihedral case.

We are going to solve this problem using the combination of two completely different tricks: first, we will prove another Lemma that will allow us to eliminate the case  $p = 2k' - 3$ , we will prove that, with our ramification conditions, such a bad-dihedral case can not happen. Then, once we are reduced to control the bad-dihedral case with  $p = 2k' - 1$ , since this case can

sometimes happen, we will isolate those cases where this equality is satisfied, and for such cases we will prove that there is another Galois conjugation, given by  $t' > t$  the first prime to  $m$  integer after  $t$ , such that by using this exponent to conjugate the new weights  $k_1$  and  $k_2$  are again smaller than the given  $k$ , and the equality  $p = 2k' - 1$  can not be satisfied by any of these two values (recall that we are assuming that it was satisfied by some of the corresponding values for  $t$ ). In other words: it is not possible that using both conjugations, the one corresponding to  $t$  and the one corresponding to  $t'$ , both give bad-dihedral cases. Thus, this extra conjugation will allow us to ensure that there is always a way to conjugate in order to reduce the weight, avoiding the bad-dihedral case.

The main difficulty is to show that  $t'$  is still sufficiently small with respect to  $m$ , so as to imply that after this conjugation the new Serre's weights are smaller than the given one.

So, the first thing to do is to prove the result that allows us to get rid of the  $p = 2k' - 3$  case. The following result is proved in [Kh] for the level 1 case, but we are going to apply it in a more general situation, where it can be seen that it still holds applying the exact same reasoning:

**Lemma 3.2** *Let  $\bar{\rho}$  be an irreducible representation in characteristic  $p$ , for a prime  $p > 2$ , that is attached to some newform  $f$ . Suppose that the Serre's weight  $k$  of  $\bar{\rho}$  satisfies  $2 \leq k \leq p+1$ . Suppose that for every prime  $w$  different from  $p$  where  $\bar{\rho}$  is ramified, the order of the image of the inertia group at  $w$  is odd. Then, the bad-dihedral case with  $p = 2k - 3$ , which corresponds to the local at  $p$  representation being induced from a ramified character of  $\mathbb{Q}_{p^2}$ , can not occur.*

Remark: With the stronger assumption that the residual representation ramifies only at  $p$ , this is Lemma 6.2 (i) in [Kh], and the proof given there extends to our case. For the reader convenience, let us briefly recall the argument. Suppose that the image of the representation is dihedral, i.e., we consider the projectivization  $\mathbb{P}(\bar{\rho})$ , and if we call  $L$  the fixed field of its kernel,  $L$  is a Galois number field with dihedral Galois group  $D_{2t}$  of order  $2t$ ,  $t > 1$ . It follows from the hypothesis that the only quadratic number field  $K$  that can be contained in  $L$  is the quadratic number field unramified outside  $p$  (in any other case, the image of inertia at some other prime would be even). From this, it follows that  $t$  is odd, because for even  $t$  dihedral groups are known to

have more than one index two subgroup (even if in general only one of them is cyclic), therefore if  $t$  were even  $L$  would contain more than one quadratic field.

This is enough to rule out the case  $p = 2k - 3$ , because in such a case locally at  $p$  the image of the projective representation is a dihedral group of order 4, but  $D_{2t}$  can not contain a subgroup of order 4 since we have shown that  $t$  is odd.

Now we resume the process of WRGC: recall that we start with a representation of some weight  $k > 16$  and level  $43^2$ , and we pick the first prime  $p$  greater than  $k$ , except for  $k = 32$  and  $k = 42$  where the value of  $p$  will be specified later. In particular, we are always taking  $p \neq 43$ .

As in Step 2, the reduction process will be guaranteed by certain inequalities that hold for  $k > 36$  due to a strong version of Bertrand's postulate, and for  $16 < k \leq 36$  and  $k = 42$  it will be checked by hand (see Step 2).

Let us start with the case  $k > 36$ ,  $k \neq 42$ . We move to characteristic  $p$  and reduce modulo  $p$ . As we already explained, Lemma 3.1 implies that this residual representation is not bad-dihedral because  $p$  is near  $k$ , more precisely:  $p/k < 1.144$ .

We take a modular weight 2 lift, whose nebentypus  $\omega^{k-2} = (\omega^d)^a$  is a character of order  $m = (p-1)/d > 6$ , where  $d = (p-1, k-2)$ . Observe that the exponent  $a$  is prime to  $p-1$ . By choosing some  $t$  relatively prime to  $m$ ,  $m/2 < t < m-1$ , which exists because  $m > 6$ , we consider the Galois conjugate of this representation having nebentypus  $(\omega^d)^t$ . As in Step 2, from the fact that  $k-2$  is near  $p-1$  (and thus dividing by  $d$ ,  $a$  is near  $m$ ) it follows that if  $t$  is taken sufficiently near  $m/2$ , thus sufficiently away from  $m$ , the largest possible Serre's weight of the residual conjugated representation,  $k_1 = dt + 2$ , will be smaller than  $k$ . In Step 2 we took  $t$  to be the smallest possible number satisfying the above conditions. Now we need to be more careful, because we want to ensure that the residual conjugated representation is not bad-dihedral. We can apply Lemma 3.2, because the only prime other than  $p$  where this residual representation ramifies is 43, and ramification at 43 is of order 11, thus we conclude that the bad-dihedral case with Serre's weight satisfying  $p = 2k' - 3$  can not happen. Therefore, we only have to deal with the case  $p = 2k' - 1$ . Observe that in all this process the Serre's weights are going to be even, because the determinant is unramified outside  $p$ , and modular Galois representations are odd. Then, in particular, if  $p \equiv 1 \pmod{4}$ , since  $2k' - 1 \equiv 3 \pmod{4}$  we are already safe and we know that



the bad-dihedral case can not happen. Later when we check by hand the weight reduction process for  $k < 38$  this will be very important: whenever we choose the prime  $p \equiv 1 \pmod{4}$  then bad-dihedral cases can not occur (in this weight reduction).

Unfortunately, it is not clear that for any gigantic  $k$  one can find a prime  $p \equiv 1 \pmod{4}$  larger than  $k$  sufficiently near  $k$ , so we still have to deal with the problem of bad-dihedral cases. The good thing is that we can restrict to the case  $p \equiv 3 \pmod{4}$ . With this restriction in mind, since  $d = (p-1, k-2)$  is always even, we see that  $m = (p-1)/d$  is odd.

Let us first use the same Galois conjugation that we used in Step 2, where  $t$  is the first prime larger than  $m/2$  coprime to  $m$ . Since in our current situation  $m$  is odd, this gives  $t = (m+1)/2$ .

Remark: in what follows, since we are restricted to the case  $m$  odd,  $m \geq 7$ , we have  $\phi(m) \geq 6$ . This implies that there exists an integer  $t'$  prime to  $m$  with:

$$m/2 < t < t' < m-1$$

We conjugate using  $t$ , we compute the two possible values of the Serre's weight  $k'$  of the residual conjugated representation:  $k_1 = dt+2$ ,  $k_2 = p+3-k_1$ , and let us check what are exactly the cases where one of these two values satisfies  $p = 2k' - 1$ . First, we observe that  $k_1 + k_2 = p+3$  and  $k_1 > k_2$ . This inequality, appearing already in [Di09], is due to the fact that  $k_1 > (p+3)/2$ , which follows easily from the fact that we are taking  $t > m/2$ . Thus, we have  $2k_1 - 1 > 2k_1 - 3 > p$ , and this implies that  $k_2 = p+1-dt$ , the smallest of the two values of the Serre's weight, is the only one that can give a bad-dihedral case.

So, let us suppose that we have:

$$p = 2k_2 - 1 = 2(p+1-dt) - 1$$

Or equivalently:

$$p+1-2dt = 0$$

Using  $p = md+1$ ,  $m = 2t-1$ , and expanding, we see that this is equivalent to:

$$((2t-1)d+1)+1-2dt = 0$$

And this is equivalent to:

$$d = 2$$

Therefore, we see that the (possibly) bad-dihedral case occurs exactly when  $p \equiv 3 \pmod{4}$ ,  $t = (m+1)/2$  and  $d = (p-1, k-2) = 2$ .

We stress that except in this specific case, the process of WRGC is done exactly as in Step 2 using the value of  $t$  as defined there, and what follows is an alternative Galois conjugation to be applied specifically in this case.

So, suppose that  $p \equiv 3 \pmod{4}$ ,  $d = (p-1, k-2) = 2$ . Then,  $m = (p-1)/2$  is odd, and as in Step 2 we know that  $m > 6$ .

As we remarked, the following integer exists: let  $t'$  be the smallest integer prime to  $m$  satisfying  $t = (m+1)/2 < t' < m-1$ . Let us use this value for conjugation instead of  $t$ . First of all, it is clear that using  $t'$  we are not going to be in a bad-dihedral case, precisely because we are assuming that using  $t$  we are in such a bad case. In fact, if both things happen at the same time we would have that in both cases  $k_2$  satisfies:  $p = 2k_2 - 1$ , thus the value of  $k_2$  would be the same for both conjugated residual representations. Replacing by the values of these weights we would obtain the equality (recall that we are assuming that  $d = 2$ )

$$p+1-2t = p+1-2t'$$

contradicting the fact that  $t \neq t'$ . In this argument we are using the fact that for  $t$  and for  $t'$  (which is larger than  $t$ ) we know that the largest Serre's weight  $k_1$  is too big to correspond to the bad-dihedral case.

Thus, it remains to check that by conjugating using  $t'$ , the new Serre's weights are smaller than the given weight  $k$ , and it is of course enough to check this for the largest of the two values,  $k_1 = 2t' + 2$ .

In order to see this, we begin by proving the following elementary Lemma:

**Lemma 3.3** *Let  $m$  be an odd integer satisfying  $m \geq 7$ .*

*Let  $p'$  be the smallest odd prime not dividing  $m$ . Then, it holds:*

$$p' < 0.6 \cdot m$$

Proof: we divide the proof in four cases:

- i)  $3 \nmid m$ : in this case,  $p' = 3$ , and since  $m \geq 7$  the Lemma follows because  $3 < 0.6 \cdot 7 = 4.2$ .
- ii)  $3 \mid m$  but  $5 \nmid m$ : in this case  $p' = 5$  and since  $m \geq 9$  the Lemma follows because  $5 < 0.6 \cdot 9 = 5.4$ .
- iii)  $15 \mid m$  but  $7 \nmid m$ : in this case  $p' = 7$  and since  $m \geq 15$  the Lemma follows because  $7 < 0.6 \cdot 15 = 9$ .

iv)  $105 \mid m$ : in this case if we have  $p' = p_{r+1}$  it holds:  $r \geq 4$  and  $m$  is divisible by the  $r - 1$  first odd primes  $p_2, p_3, \dots, p_r$ . Then, because by Bertrand's postulate  $p' < 2 \cdot p_r$  we obtain:

$$p' < 9 \cdot p_r = 0.6 \cdot 3 \cdot 5 \cdot p_r = 0.6 \cdot p_2 \cdot p_3 \cdot p_r \leq 0.6 \cdot p_2 \cdot p_3 \cdot \dots \cdot p_r \leq 0.6 \cdot m$$

and this concludes the proof.  $\blacksquare$

We have defined  $t'$  to be the smallest integer that is prime to  $m$  and larger than  $(m + 1)/2$ . Then, it is easy to see that if we define  $p'$  as in the Lemma above, it holds:

$$t' = (m + p')/2$$

Therefore, applying Lemma 3.3 we conclude that  $t' < 0.8 \cdot m$ .

From this, we have:  $k_1 = 2t' + 2 < 1.6m + 2$ , and since  $m = (p - 1)/2$  this gives:

$$k_1 < 0.8 \cdot (p - 1) + 2 = 0.8 \cdot p + 1.2$$

On the other hand, we are assuming that  $k > 36$  in order to use, as in Step 2, the validity of a strong version of Bertrand's postulate that gives:  $p < 1.144 \cdot k$ . Combining this inequality with the previous one we obtain:

$$k_1 < 0.8 \cdot 1.144 \cdot k + 1.2 = 0.9152 \cdot k + 1.2$$

Also, since  $k > 36$ , it is easy to see that  $0.9152 \cdot k + 1.2 < k$ , and then we conclude that  $k_1 < k$ , which concludes the proof of the induction, for  $k > 36$  and  $k \neq 42$ .

It remains to see that for the remaining values of  $k$  either  $t$  defined as in Step 2 or  $t'$  as defined above makes the induction work. We will list all cases, starting from the special values  $k = 42$  and  $k = 32$ .

In all cases  $p \neq 43$ ,  $p \neq 2k - 1$ , and also  $k_1, k_2 < k$ . Moreover, since we are using  $t'$  instead of  $t$  precisely in those cases where  $p \equiv 3 \pmod{4}$  and  $d = 2$ , the equality  $p = 2k' - 1$  for  $k' = k_1$  or  $k_2$  is never satisfied:

$$k = 42, p = 47 : d = 2, m = 23, t' = 13, k_1 = 28, k_2 = 22$$

$$k = 32, p = 47 : d = 2, m = 23, t' = 13, k_1 = 28, k_2 = 22$$

$$k = 36, p = 37 : d = 2, m = 18, t = 11, k_1 = 24, k_2 = 16$$

$$k = 34, p = 37 : d = 4, m = 9, t = 5, k_1 = 22, k_2 = 18$$

$$k = 30, p = 31 : d = 2, m = 15, t' = 11, k_1 = 24, k_2 = 10$$

$$k = 28, p = 29 : d = 2, m = 14, t = 9, k_1 = 20, k_2 = 12$$

$$k = 26, p = 29 : d = 4, m = 7, t = 4, k_1 = 18, k_2 = 14$$

$$k = 24, p = 29 : d = 2, m = 14, t = 9, k_1 = 20, k_2 = 12$$

$$k = 22, p = 23 : d = 2, m = 11, t' = 7, k_1 = 16, k_2 = 10$$

$$k = 20, p = 23 : d = 2, m = 11, t' = 7, k_1 = 16, k_2 = 10$$

$$k = 18, p = 19, d = 2, m = 9, t' = 7, k_1 = 16, k_2 = 6$$

$$k = 16, p = 17 : d = 2, m = 8, t = 5, k_1 = 12, k_2 = 8$$

This concludes the WRGC process, and we end up with a newform  $f_9$  of level  $43^2$  and weight  $2 < k \leq 14$ .

### 3.7 Step 7

This step is meant to reduce to cases where  $k < 17$  and divisible by 4. This last condition will be required in Step 8.

We start with  $f_9$  and the idea is, as in Step 3, to unify the value of the weight using the Sophie Germain trick and then, since the weight obtained is too big, to perform some extra weight reductions, in a controlled way. This time we will apply Khare's method of weight reduction (cf. [Kh]).

To preserve the MGD prime 43 in the level, we are going to avoid characteristics 11 and 43 at this Step.

Once again, as explained at the beginning of Step 5, in order to ensure that residual images are 6-extra large it is enough to check that they are not dihedral.

So, let us first choose the pair of Sophie Germain primes 23 and 47. We consider a mod 47 Galois representation attached to  $f_9$ . Since ramification at 43 has odd order, the only possible dihedral case is the bad-dihedral case. Since the residual Serre's weight satisfies  $2 < k \leq 14$  and  $p = 47$  it follows from Lemma 3.1 that the representation can not be bad-dihedral, thus it can not be dihedral.

We take a modular weight 2 lift, with nebentypus  $\omega^{k-2}$ ,  $2 < k \leq 14$ , we move to characteristic 23, and we reduce mod 23. Observe that this residual representation has Serre's weight 2 and is ramified at most at 47, 23 and 43. Ramification at 47, since we are reducing mod 23 a character of order 23, can either be trivial or unipotent. In any case, if we suppose that the residual

representation has dihedral image, it must be unramified at 47. Again, since 43 is a MGD prime, the only possible dihedral case is the bad-dihedral case, which is ruled out by Lemma 3.1 because  $k = 2$ .

We take a lift of weight 2 of this residual representation, corresponding to a newform that is Steinberg locally at 47 (in case the residual representation is unramified at 47, we are raising the level: as in step 3, it is easy to see that the necessary condition for level raising is satisfied). We move to characteristic 47 and we reduce mod 47. This residual representation can only have dihedral image if it has bad-dihedral image (the only ramified prime other than 47 being the MGD prime 43), which is ruled out again by Lemma 3.1 because the Serre's weight is  $k = 48$  or  $k = 2$ .

In any case, we can take an ordinary lift of this residual representation of weight 48 and level  $43^2$  (the specialization at  $k = 48$  of the Hida family containing the weight 2, Steinberg at 47 newform). Thus, we have been able to reduce to a case of weight 48. Even if 48 is divisible by 4, it is too big for our purposes, so let us continue by linking this representation with some other representation of smaller weight.

At this point, we apply a particular case of Khare's weight reduction: we pick a prime  $p$  larger than  $k$ , we reduce modulo  $p$  and take a modular weight 2 lift, and we also select an auxiliary prime  $r$  dividing  $p - 1$  so that by reducing modulo  $r$  the order of the nebentypus changes. This way, one can take a lift of this mod  $r$  representation with a different type at  $p$ , in our case we are taking a lift that is minimal at  $p$ , and going back to characteristic  $p$  the value of the residual Serre's weight will be different than the initial one (see [Kh] for details).

We have a newform of weight 48 (and level  $43^2$ ), and we choose  $p = 53$ . We reduce mod 53 (as usual, the residual image is not dihedral because of Lemma 3.1) and take a modular weight 2 lift with nebentypus  $\omega^{46}$  of order 26.

We move to  $r = 13$  and we reduce modulo 13. This residual representation has weight 2, and it ramifies at 13, 43 and 53. Since ramification at 43 has odd order, and applying Lemma 3.1, we conclude that if the image is dihedral, it must be the case that the representation is induced from the quadratic number field ramified only at 53.

Luckily, the MGD prime 43 is a square modulo 53, then we can argue as with Good-Dihedral primes (this is the argument used to obtain large images with Good-Dihedral primes, as in [KW]) to get a contradiction. Let  $K = \mathbb{Q}(\sqrt{53})$ . Suppose that the residual representation is induced from a character of  $G_K$ ,

in which case its restriction to  $G_K$  is reducible. Since 43 is split in  $K$ , the image of the decomposition group at 43 is contained in the restriction to  $G_K$  of the representation, and this gives a contradiction since the first one is irreducible because 43 is a MGD prime and the latter is reducible.

This proves that the mod 13 representation can not have dihedral image.

Before reducing modulo 13, we had a nebentypus of order 26. Thus, if we take a minimal lift of this mod 13 representation, it will correspond to a weight 2 newform of level  $43^2 \cdot 53$  but this time ramification at 53 will be given by a character of order  $26/13 = 2$ , namely,  $\omega^{26}$ .

We go back to characteristic 53, and the possible values for the Serre's weight of this residual representation are  $26 + 2 = 28$ , or  $53 + 3 - 28 = 28$ , so  $k = 28$ . This implies that the image is not bad-dihedral, because even if  $53 = 2 \cdot 28 - 3$  we can apply Lemma 3.2 (the only prime in the level is the MGD prime 43). Therefore, the residual image is not dihedral.

So far, we have reduced to the case of a newform of weight  $k = 28$  and level  $43^2$ . Now we apply again Khare's weight reduction, taking  $p = 29$ . We reduce mod 29 (thanks to Lemma 3.1 the residual image is not dihedral) and take a modular weight 2 lift, whose nebentypus is  $\omega^{26}$ , a character of order 14.

We move to  $r = 7$ , and obtain a mod 7 representation  $\bar{\rho}$  of weight 2 ramified at 7, 43 and 29. If we suppose that this representation is dihedral, induced from a character of  $G_K$  for some quadratic number field  $K$ , then due to lemma 3.1 we know that  $K$  can not ramify at 7, and since 43 is a MGD prime we conclude that it must be  $K = \mathbb{Q}(\sqrt{29})$ .

It takes some effort to show that this is not possible: in fact after reduction modulo 7 ramification at 29 is given by a character of order  $14/7 = 2$ , so a priori  $\bar{\rho}$  could be induced from  $K$ . The key to show that this is not possible is that  $K$  is real, and 7 is split in  $K$ . Supposing that the restriction to  $G_K$  of  $\bar{\rho}$  is reducible, then using the fact that the Serre's weight is 2 and that 7 is split in  $K$ , the first thing that we notice is that the restriction of  $\bar{\rho}$  to the inertia group  $I_7$  must be given by the sum of the characters  $\chi$  and 1.

Then, we compute using PARI/GP the ray class field of  $K$  of conductor  $\hat{7}$ , for  $\hat{7}$  a prime in  $K$  dividing 7 and we see that it gives an extension of  $K$  of order 12, which is given by the composition of the cyclotomic field of 7-th roots of unity and the quadratic extension corresponding to a square root of (a generator of the ideal corresponding to) the prime  $\hat{7}$ . Let us call  $\psi$  the quadratic character of  $G_K$  corresponding to the latter.

Let  $\mu_1, \mu_2$  be the two mod 7 characters whose sum is isomorphic to the re-

striction to  $G_K$  of  $\bar{\rho}$ , then we have (in some order):

$\mu_1 = \chi^i \psi^j \phi$ ,  $\mu_2 = \chi^{1-i} \psi^j \phi^{-1}$ , with  $1 \leq i \leq 5$  and  $j = 0$  or  $1$ , where  $\phi$  is unramified outside  $43$ .

Combining this information with the local information at inertia at  $7$  already mentioned it is easy to see that the following must hold:  $j = 0$  and  $i = 1$ . But this contradicts the fact that this is the restriction of an induced representation, because in such a case the two characters should be inner conjugates of each other.

This concludes the proof that the mod  $7$  representation does not have dihedral image.

We now proceed as in the previous reduction: we take a minimal lift of this residual representation, observing that now the ramification at  $29$  is given by the quadratic character  $\omega^{14}$ .

We go back to characteristic  $29$ , where the residual Serre's weight is  $14 + 2 = 16$  or  $29 + 3 - 16 = 16$ , and the residual image is not bad-dihedral (thus not dihedral) due to Lemma 3.2.

We end up this step with a newform  $f_{10}$  of weight  $16$  and level  $43^2$ .

### 3.8 Step 8

This is a very elementary step, we have isolated this move just because of its conceptual importance.

Since the weight of  $f_{10}$  is  $16$ , we can move to characteristic  $17$  and produce a congruence with a weight  $2$  newform  $f_{11}$  of level  $43^2 \cdot 17$ , whose ramification at  $17$  is given by the character  $\omega^{14}$  of order  $8$ , a character that is also the nebentypus of this newform.

In this congruence, the residual representation has Serre's weight  $16$  and  $p = 17$ , then an application of Lemma 3.1 together with the fact that  $43$  is a MGD prime (where ramification has an odd order) proves that the residual image is not dihedral. Due to the MGD prime in the level, this is enough to conclude that the residual image is 6-extra large (see discussion at beginning of Step 5).

This Step concludes with  $f_{11}$ , a weight  $2$  newform with nebentypus of order  $8$  at  $17$  and the MGD prime  $43$  in the level.

### 3.9 Step 9

At this step we are going to remove the MGD prime 43 from the level of  $f_{11}$ , more precisely we will transform it into a Steinberg prime via a modulo 11 congruence.

The problem is that since we are losing the MGD prime it is not clear a priori that the residual image will be irreducible in this congruence. Moreover, the space of newforms of weight 2 and level  $43^2 \cdot 17$  seems to be too big for computations, so we can not check by hand that a mod 11 representation attached to  $f_{11}$  is irreducible.

Luckily, the local information of the 11-adic Galois representation at the ramified primes 43 and 17 is enough to deduce that the residual representation is irreducible, as proved in the following:

**Lemma 3.4** *Consider the weight 2 newform  $f_{11}$ , whose level is  $43^2 \cdot 17$ , with nebentypus  $\psi$  of order 8 ramified at 17 and such that 43 is a MGD prime with ramification at 43 having order 11. Then, any modulo 11 residual representation attached to  $f_{11}$  is irreducible.*

Proof: Let  $\bar{\rho}$  be a residual representation in characteristic 11 attached to  $f_{11}$ . Suppose that  $\bar{\rho}$  is reducible, and let us call  $\mu_1, \mu_2$  the two characters in the diagonal. Since 43 was a supercuspidal prime, and ramification at 43 was given by a character of order 11, it is well-known that this residual representation will either be unramified or have unipotent ramification at 43. In any case, it is clear that the characters  $\mu_1$  and  $\mu_2$  are unramified at 43. From now on, we consider the semi-simplification of the residual representation, which is isomorphic to the direct sum  $\mu_1 \oplus \mu_2$ . Using the information on the ramification at 17, and the fact that the newform is of weight 2 and level prime to 11, we conclude that there are just two possibilities for this direct sum: it must be either  $\chi\psi \oplus 1$  or  $\chi \oplus \psi$ .

On the other hand, the 11-adic lift of  $\bar{\rho}$  provided by  $f_{11}$  is, locally at 43, induced from a character of the unramified quadratic extension of  $\mathbb{Q}_{43}$ , and therefore the residual representation must satisfy the trace 0 condition necessary for the existence of such a lift: in terms of the characters  $\mu_i$  the condition is:

$$\mu_1(43) + \mu_2(43) = 0$$

(recall that this is an equality in characteristic 11).

We plug into this formula the two possible values for these characters and in



both cases we obtain (again in characteristic 11):

$$\psi(43) = 1 \quad (@)$$

Since  $\psi$  is a character ramified at 17 of order 8, if we take a prime  $w$  which is a primitive root modulo 17 the value  $\psi(w)$  will be an element of order 8 in some extension of  $\mathbb{F}_{11}$ . An easy computation shows that the order of 43 modulo 17 is 8, thus  $43 \equiv w^2 \pmod{17}$  for some primitive root  $w$ . Therefore,  $\psi(43) = \psi(w^2) = \psi(w)^2$  gives an element of order 4 in some extension of  $\mathbb{F}_{11}$ . But this clearly contradicts (@), and this concludes the proof that  $\bar{\rho}$  is irreducible.

■

We have thus seen that the mod 11 representation  $\bar{\rho}$  attached to  $f_{11}$  is irreducible. Let us check, using once again Dickson's classification, that its image is large. First of all, the projective image can not be an exceptional group because the image of ramification at 17 gives an element having, even after projectivization, order 8.

Since the Serre's weight is 2, it follows from Lemma 3.1 that, if the image is dihedral, it must be induced from a quadratic field  $K$  ramifying at most at 17 and 43. But ramification at 43 of the residual representation is either trivial or unipotent (thus in both cases of odd order), so  $K$  can only ramify at 17.

To show that this case can not occur, we argue as in the proof of Lemma 3.1. We are now considering ramification at a prime different from the residual characteristic, but nevertheless the same reasoning on dihedral groups applies: If we assume that the image of  $\bar{\rho}$  is dihedral induced from  $K$  and at a ramified prime  $r$  we know that the image of the inertia group is given by a cyclic group of projective order greater than 2, then the quadratic field  $K$  can not ramify at  $r$ .

Thus, we conclude that the residual image can not be dihedral. Therefore, from Dickson's classification we see that the image is large. Moreover, since  $\mathrm{PGL}(2, \mathbb{F}_{11})$  does not contain elements of order 8, we see (using a generator of the inertia group at 17) that the image is 6-extra large.

As we have already remarked, the residual representation  $\bar{\rho}$  will have either trivial or unipotent ramification at 43. Moreover, in the unramified case, it is easy to see from the existence of the lift given by  $f_{11}$ , which is supercuspidal at 43, that the necessary condition for the existence of a lift that is Steinberg at 43 is satisfied. In fact, since  $43 \equiv -1 \pmod{11}$ , this condition simply reads  $\bar{\rho}(\mathrm{Frob}_{43}) \equiv 0 \pmod{11}$ , and we have already stressed during

the proof of Lemma 3.4 that this is satisfied. In any case, *soit* by Steinberg level-raising, *soit* by taking a minimal lift, we conclude that there is a modular lift of  $\bar{\rho}$  corresponding to a weight 2 newform  $f_{12}$  of level  $43 \cdot 17$ , which is Steinberg at 43 and has a nebentypus of order 8 at 17.

### 3.10 Step 10

What we do now is to move to characteristic 43 and consider the residual representation attached to  $f_{12}$  in this characteristic. In order to show that this residual representation is irreducible, we perform a computation using MAGMA in the full space of newforms of weight 2 and level  $43 \cdot 17$ , with nebentypus given by any character  $\psi$  of order 8 ramified at 17. The output of this computation is that any residual representation in characteristic 43 attached to any of these newforms is irreducible.

It took us only a few minutes to check this, because we used some theoretical information to speed the computation. In fact, all we did was to compute the Hecke polynomial  $P_2(x)$  of the Hecke operator  $T_2$  in the above mentioned space. Then, we studied what are the possible characters that would appear in the reducible case, i.e., if the residual representation is assumed to be reducible, we call  $\mu_1$  and  $\mu_2$  the characters in the diagonal, and using local information we conclude that their sum will be either  $\chi\psi \oplus 1$  or  $\chi \oplus \psi$ , as in the previous step. In any of the two cases, we evaluate this sum at the prime 2 and we consider  $Q(x)$  to be the minimal polynomial of the value obtained. The two different values for  $Q(x)$  that we obtain are:

$$(x - 2)^8 - 1, \quad \text{and} \quad (x - 1)^8 - 2^8$$

To conclude, we simply check using resultants that the polynomials  $P_2(x)$  and  $Q(x)$  (we consider here the two possible values of the latter) do not have any common root modulo 43.

Knowing that the modulo 43 representation is irreducible, let us check that its image is 6-extra large. Again, the ramification at 17 makes easy to see that the image can not be exceptional, and for the dihedral case we just observe that since the Serre's weight is either 2 or 44 it can not be dihedral corresponding to a field  $K$  ramifying at 43 due to Lemma 3.1, and as in the previous step we also eliminate the case where  $K$  ramifies at 17 because the image of the ramification group at 17 contains an element of projective order

8.

We conclude from Dickson's classification that the image is 6-extra large.

We consider a modular lift of this residual representation, corresponding to a newform  $g$  of weight 44 and level 17, with nebentypus at 17 given by a character  $\psi$  of order 8. Observe that both  $f_{12}$  and  $g$  are ordinary at 43, they live in the same Hida family.

### 3.11 Step 11

This step consists just on a computation performed using MAGMA on the space of newforms of weight 44 and level 17, with any nebentypus of order 8 at 17. Recall that  $g$ , the final output of our chain, lives in this space. In agreement with some expectations based on generalizations of Maeda's conjecture (work in progress of P. Tsaknias jointly with the author), we conjectured that this space contains a unique orbit of Galois conjugated newforms.

This was confirmed by our MAGMA computation. This was computed by P. Tsaknias using the MAGMA command `NewformDecomposition`, it took 14 hours in a CPU Intel Xeon E7-4850, 2GHz, with 192GB RAM. The computer used was the server of the Number theory Group of The Department of Mathematics of the University of Luxembourg.

Therefore, we have managed to link any given level 1 cuspform  $f$  with the orbit of  $g$ , and the chain constructed is safe, in the sense that current A.L.T. (including the result in Appendix B) can be applied in both directions, at any link, to the 5-th symmetric powers of the Galois representations considered. In particular, by transitivity, together with the fact that Galois conjugation is a valid move (so that we can move freely in the orbit of  $g$ ), any given pair of level 1 cuspforms can be linked to each other in a safe way. Thus, using our chain, we conclude that automorphy of  $\text{Symm}^5(f)$  will hold for any level 1 cuspform  $f$  provided that we can prove it for a single example.

## 4 The base case for the automorphy of $\text{Symm}^5(\text{GL}(2))$

For the base case, we start by considering an example of a weight 1 cuspform with projective image  $A_5$  studied in [KiW]. This example consists on a cuspform  $f'_0$  of prime level 2083, weight 1, and quadratic nebentypus. We can consider this complex representation as taking values on a finite field of

characteristic 2083, and the obtained residual representation will have Serre's level 1 and Serre's weight 1042, so we know that there is a level 1 cuspform  $f_0$  of weight 1042 such that this residual representation is attached to it. Our goal is to prove automorphy for  $\text{Symm}^5(f_0)$ , by exploiting its congruence with  $\text{Symm}^5(f'_0)$ .

Using results of Kim and Wang (cf. [Kim], [W]), it is known that  $\text{Symm}^5(f'_0)$  is cuspidal and automorphic. If we call  $\rho$  the complex Galois representation attached to  $f'_0$ , this uses the identity:

$$\text{Symm}^5(\rho) \cong \text{Symm}^2(\rho') \otimes \rho \quad (*)$$

where  $\rho'$  denotes the Galois conjugate representation of  $\rho$ , which has trace defined over  $\mathbb{Q}(\sqrt{5})$ .

But this is not enough to conclude residual automorphy of the 2083-adic Galois representation attached to  $f_0$ , in the sense required for the application of A.L.T., because the automorphic form attached to  $\text{Symm}^5(f'_0)$  is clearly not regular.

In order to solve this problem we first consider  $\bar{\rho}$  and  $\bar{\rho}'$ , the mod 2083 representations obtained from  $\rho$  and  $\rho'$ , and we replace them in formula (\*). Observe that since 2083 does not divide the order of the image the formula in characteristic 2083 is still an equality between irreducible Galois representations. We now consider lifts of the residual representations  $\bar{\rho}$  and  $\bar{\rho}'$  attached to cuspforms of weights greater than 1. More precisely, we know that can take a 2083-adic Galois representation attached to  $f_0$ , a cuspform of weight 1042, as a lift of  $\bar{\rho}$ , and we can take some 2083-adic Galois representation attached to a weight 2 newform  $f_1$  minimally lifting  $\bar{\rho}'$  (it exists because any mod  $p$  odd, irreducible, representation has a weight 2 minimal modular lift). Since  $\bar{\rho}'$  also has ramification at 2083 given by a quadratic character, it is clear that  $f_1$  has level 2083 and that its 2083-adic Galois representations are potentially Barsotti-Tate.

By plugging the representations attached to  $f_0$  and  $f_1$  into the right hand side of formula (\*), and the one attached to  $f_0$  on the left hand side, we conclude that there is a modulo 2083 congruence between the Galois representations attached to  $\text{Symm}^5(f_0)$  and to  $\text{Symm}^2(f_1) \otimes f_0$ .

As in the work of Kim and Wang, we can deduce from known cases of Langlands functoriality (the  $\text{Symm}^2(\text{GL}(2))$  case due to Gelbart and Jacquet in [GJ] and the  $\text{GL}(2) \times \text{GL}(3)$  case due to Kim and Shahidi in [KS]) that the latter tensor product is cuspidal automorphic (cuspidality follows from the criterion given in section 2 of [W]). Moreover, from our choice of the weights

of  $f_0$  and  $f_1$  it is clear that its attached Galois representations are regular, and using the fact that for  $f_0$  the prime 2083 is not in the level and is in the Fontaine-Laffaille range and that  $f_1$  is potentially Barsotti-Tate at 2083 we see that the 2083-adic Galois representation attached to the automorphic form  $\mathrm{Symm}^2(f_1) \otimes f_0$  is potentially diagonalizable.

On the other hand, we also know that the 2083-adic Galois representation attached to  $\mathrm{Symm}^5(f_0)$  is potentially diagonalizable (again, because 2083 is in the Fontaine-Laffaille range for  $f_0$ ). Finally, observe that the residual projective image in the congruence between 6-dimensional Galois representations that we are considering is irreducible and isomorphic to  $A_5$ , thus clearly the residual representation will stay irreducible over any cyclotomic extension, thus it follows from the main result of [GHTT] that its image is adequate even after restriction to a cyclotomic field.

We have all the ingredients to apply the A.L.T. in [BLGGT] to deduce automorphy of  $\mathrm{Symm}^5(f_0)$  from this mod 2083 congruence, because both 6-dimensional Galois representations are regular and potentially diagonalizable, and the residual image satisfies the required condition. Therefore, the automorphy of  $\mathrm{Symm}^2(f_1) \otimes f_0$  implies that of  $\mathrm{Symm}^5(f_0)$ . This concludes the proof of the base case and, due to the results of the previous section, we also conclude automorphy of  $\mathrm{Symm}^5(f)$  for any given level 1 cuspform  $f$ .

## 5 Application to base change

A “straightforward” concatenation of the safe chain constructed in Section 3 with the process of “killing ramification” (including ramification swapping) as used in section 4 of the paper [Di12b] gives a new proof of base change for  $\mathrm{GL}(2)$ , for any newform of odd level, this time without local conditions on the totally real number field  $F$ . We do not even assume that  $F$  is a Galois number field. Thus, this proof works in much more general situations than the one given in [Di12a]: it should be observed that we are relying on the construction given in section 3 and this construction owes a lot to the methods created in [Di12a].

Moreover, applying the 2-adic Modularity Lifting Theorem in [K-2] together with an adaptation of ideas from [KW] we can extend the proof of base change to newforms of arbitrary level by killing ramification at 2 without increasing the weight.

We begin by explaining the proof for newforms of odd level, we will explain

the case of even level at the end of this section. As usual, adding the Good-Dihedral prime as in step 1 in section 3 is done first, before anything else, and the primes  $t$  and  $q$  are taken larger than a bound  $B$  that is itself larger than any other auxiliary prime to appear in the rest of the proof (this is standard, compare with [Di12b] and [Di12a]).

The base case for this new proof will be a CM modular form of odd level, and the fact that the part of the chain that we are going to take from [Di12b] is good enough to propagate modularity over  $F$  in both directions can be easily checked: residual images are large because of a Good-Dihedral prime  $q$ , this is true in particular when the characteristic is 3 or 5 because the projective image of the inertia group at  $q$  gives a cyclic group whose order  $t$  is a very large prime (and largeness is preserved by base change because the representations are odd and  $F$  is totally real, cf. [Di12a], Lemma 3.2. In fact, the easiest way to see this is to pass first to the Galois closure  $\hat{F}$  of  $F$ : since the image of the restriction to  $G_{\hat{F}}$  contains the image of complex conjugation and is a normal subgroup of the full image, we easily see that the image of this restriction is large, thus *a fortiori* the image of the restriction to  $G_F$  is also large), and the two main A.L.T. in [BLGGT] (and the result in Appendix B, cf. [DG]) apply during the process of ramification swapping (there are two rounds of swapping, described in Lemmas 4.1 and 4.2 in [Di12b]: in each congruence either both representations are ordinary or they are both potentially diagonalizable, the latter case involving potentially Barsotti-Tate and crystalline Fontaine-Laffaille representations, as in previous sections, except when the residual Serre's weight of a potentially Barsotti-Tate representation happens to be  $p + 1$ : in this case the minimal crystalline lift of weight  $p + 1$  is outside the Fontaine-Laffaille range, but is known to be ordinary, thus potentially diagonalizable), whilst for the killing ramification, as in Prop. 4.3 in [Di12b], one relies on the Modularity Lifting Theorem (M.L.T.) in [K] (observe that this theorem works well in both directions). The latter applies exactly as in [Di12b] for trivial reasons provided that one makes sure that it is applied at primes that are split in  $F$  (we are using the fact that we know that the image of the restriction to  $G_F$  is large). It is easy to see that the set of auxiliary primes appearing in the level (after the application, in two rounds, of ramification swapping) can be required to be totally split in  $F$  because this is perfectly compatible with the other condition imposed to these primes  $q_i$  in [Di12b]. In fact, when the prime  $b_i$ , a pseudo Sophie Germain prime, introduced to the level in the first round of swapping, is Steinberg (cf. Lemma 4.1 in loc. cit.), then the only condition on the prime  $q_i$  (such that

we swap ramification from  $b_i$  to  $q_i$  in the second round) imposed in loc. cit. is that it must be congruent to 1 modulo  $2 \cdot (b_i - 1)$  (cf. Lemma 4.2 in loc. cit.). When, for the pseudo Sophie Germain prime  $b_i$  introduced to the level in the first round of ramification swapping, ramification at  $b_i$  turns out to be given by a quadratic character (the second case in the output of Lemma 4.1 of loc. cit.), the second round is omitted in loc. cit. (cf. Lemma 4.2 in loc. cit.). Since it would be hard to ensure that such primes are split in  $F$ , we can simply proceed to apply the second round of ramification swapping also to these primes, exactly in the same way that it is applied at the Steinberg primes in Lemma 4.2 in loc. cit.: since it is known that with this quadratic ramification at  $b_i$  a weight 2 family will have residual Serre's weight at  $b_i$  equal to  $k = (b_i + 3)/2$ , we must choose a large auxiliary prime  $q_i$  such that  $q_i \equiv 1 \pmod{2 \cdot (k - 2)}$ , i.e.,  $q_i \equiv 1 \pmod{b_i - 1}$ , and we can proceed to “swap” ramification from  $b_i$  to  $q_i$  as done with Steinberg primes in Lemma 4.2 in loc. cit., the proof via results of Caruso that the condition required to apply [K] at the killing ramification process is satisfied (cf. Prop. 4.3 and Lemma 4.4 in loc. cit.) extends verbatim to this extra case, and now we can safely say that **all** the auxiliary primes in the level after the second round of swapping can be taken to be split in  $F$ .

These remarks being done, the proof of base change proceeds as a simple concatenation: after introducing the Good-Dihedral prime  $q$  to the odd level of the given newform  $f$  in the usual way, apply the killing ramification as in [Di12b] to reduce to a “level one” situation. Actually this is not truly level 1 because the Good-Dihedral prime  $q$  is already in the level, but in any case at this point we connect with the safe chain constructed in Section 3, just observe that we start from Step 2 instead of Step 1.

In any case, it is the safe chain constructed in section 3 the one that allows us to link any pair of newforms to each other, we are just adding two tails to this chain, corresponding to the killing ramification for the given newform  $f$  at one end of the chain and for the base case form  $f_0$  at the other end. The fact that the chain from section 3 can be used to propagate modularity (in both directions) after restriction to  $G_F$  is due to the fact that the local conditions required to apply the two main A.L.T in [BLGGT] and the improvement in Appendix B are preserved by arbitrary base change, and that the largeness of the residual images is also preserved by restriction to  $G_F$ , as we already explained (and we have shown in section 3 that in this chain residual images are always 6-extra large, thus in particular they are large). The newform  $f$  is assumed to be non-CM, because base change is known in the CM case.

The only non-trivial comment to be done is that for the arguments in Section 3 we require that the initial weight be greater than 2, and it can be the case that after concluding the killing ramification one finishes with a weight 2 newform. If this happens, this can easily be remedied by acting as if the Good-Dihedral prime  $q$  in the level were not a Good-Dihedral prime, then choosing and adding another prime  $q'$  much bigger such that it is Good-Dihedral for all characteristics up to some bound  $B'$  larger than  $q$ . Then, removing  $q$  from the level in two steps (as done in Section 3), we know that we will end up with a representation of weight  $q + 1 > 2$  and level  $q'^2$ , good-dihedral at  $q'$ . In other words: a simple iteration of the argument allows us to work “as if” after the completion of the killing ramification step the weight obtained “will be” greater than 2.

Concerning the base case, it is easy to see that given any form  $f_0$  attached to a CM elliptic curve  $E$  of odd conductor, if we call  $K$  the field such that  $E$  has CM by an order of  $K$ ,  $f_0$  can be base changed to any totally real number field  $F$  (as it is well-known, since  $K$  can not be contained in  $F$  the restrictions to  $G_F$  of the  $\ell$ -adic Galois representations attached to  $f_0$  are absolutely irreducible). Also, given  $F$  we easily see that there is a prime  $p > 5$  (depending on  $F$ ) not in the level of  $f_0$  such that the residual mod  $p$  representation attached to  $f_0$  (it is of course dihedral) will be irreducible even after restriction to  $F(\zeta_p)$  (we just need to ensure that  $\hat{F}(\zeta_p)$  does not contain  $K$ , where  $\hat{F}$  denotes the Galois closure of  $F$ , but it is easy to see that this can only fail for finitely many primes).

Then, modulo such a prime  $p$  the first thing that we do is to apply level-raising to add a Steinberg odd prime  $w$  to the level (keeping the weight equal to 2). At this mod  $p$  congruence, it follows from the result in [GHTT] that the residual image restricted to  $F(\zeta_p)$  is adequate (and both representations are Barsotti-Tate), thus we see that modularity over  $F$  propagates well. After introduction of the Steinberg prime we already have a weight 2 newform with generically large residual images (cf. [Ri85]), and we can proceed to introduce the Good-Dihedral prime to the level and go on with the killing ramification. Except at this first step where the residual image is dihedral (but not bad dihedral), all the residual representations in the rest of the chain will be large.

This concludes the the proof of base change for odd level newforms where, thanks to the fact that the local conditions in the A.L.T. in [BLGGT] and in Appendix B are preserved by base change, we do not need to impose any



local condition on the field  $F$ .

Let us now focus on the case of a newform  $f$  of even level. As usual, we assume that  $f$  is a non-CM form. The idea to prove base change for such a newform is to eliminate the prime 2 from the level as in [KW], relying on the 2-adic M.L.T in [K-2], thus reducing the proof to the odd level case, already solved.

First of all, since we are going to need the 2-adic M.L.T. over  $F$ , let us observe that we are going to apply Theorem 0.9 in [K-2], but with the local condition at the prime 2, numbered as condition (3) in loc. cit., replaced by the condition:

(3') Let  $v$  be any prime of  $F$  dividing 2 such that the representation  $\rho$  restricted to the decomposition group at  $v$  is ordinary. Then the 2-adic representation  $\rho_h$ , which is attached to a Hilbert newform  $h$  and is congruent to  $\rho$  modulo 2, is also ordinary locally at  $v$ .

The fact that Theorem 0.9 in loc. cit. holds with this modification can be seen by inspecting its proof (the same happens with the M.L.T. in odd residual characteristics proved in [K-BT], where a version of the main theorem with this more general condition is recorded), it could have been stated this way.

We proceed with  $f$  as we did in the odd level case, the only difference is that at some point we are going to eliminate the prime 2 from the level. The right moment for this is after the swapping ramification process (applied to all ODD primes in the level), and before starting the killing ramification. At this point the newform, let us call it  $f'$ , contains in its level the prime 2, plus a set of relatively large auxiliary primes  $q_i$ , and a very large prime  $q$ , the good-dihedral prime, where ramification has order  $t$ , another very large prime number. Observe also that at this point of the argument  $f'$  has weight 2 (cf. [Di12b]).

Because of the Good-Dihedral prime, we know that if we work modulo a small odd characteristic the residual image will be adequate. Also, as in [KW] or [Di12a], the prime  $q$  is supposed to satisfy certain conditions with respect to the prime 2, and the prime  $t$  is taken greater than 5, and using this it follows (cf. [KW]) that if we work modulo 2 the residual projective images will be non-solvable and not isomorphic to  $A_5$ .

Let us also assume, as in [Di12a], that the prime  $q$  is split in the Galois closure  $\hat{F}$  of  $F$ , this way after restricting to  $G_{\hat{F}}$  a 2-adic Galois representation we know that the residual projective image still is absolutely irreducible and

contains an element of order  $t$ , and from this we conclude, as in [Di12a], Lemma 3.4, that the image of this restriction is again non-solvable and not isomorphic to  $A_5$ . We easily conclude that the same holds for the image of the restriction to  $G_F$ .

Let us now indicate the moves, taken from [KW], that will allow us to link the compatible system attached to  $f'$  with another corresponding to a newform of odd level, in such a way that, after restricting to  $G_F$ , modularity propagates well from the latter to the former.

We divide in two cases:

- (i) the 2-adic Galois representation attached to  $f'$  is potentially Barsotti-Tate,
- (ii) the 2-adic Galois representation attached to  $f'$  is potentially semistable of weight 2.

Let us start with case (ii). In this case, we can twist the representation by a suitable character in order to reduce to the semistable case, so let us assume that the 2-adic representation attached to  $f'$  is semistable. As in [KW], section 9, we reduce modulo 3 to change the local type at 2, preserving the weight and the ramification at all other primes (this is based on Theorem 5.1 of loc. cit.): since 3 is not in the level of  $f'$  (the primes  $q_i$  are large primes) this is just a congruence between Galois representations that are both Barsotti-Tate at 3, and because of the good-dihedral prime the residual image is large, thus adequate, therefore we know from the A.L.T. in [BLGGT] (its variant in Appendix B) that the new Galois representation we are creating is also modular, attached to certain newform  $f_2$ . This new Galois representation constructed in [KW] has a different type at 2,  $f'$  was Steinberg while  $f_2$  is principal series at 2 (ramification being given by characters of order 3). It is important to observe that the 2-adic Galois representation attached to  $f_2$  is potentially Barsotti-Tate. Moreover, it is shown in [KW] that the residual modulo 2 Galois representation attached to  $f_2$  has Serre's weight 2. Therefore, applying again Theorem 5.1 of loc. cit. (recall that we know that the projective residual image is non-solvable and not an  $A_5$ ) we can take a lift of this mod 2 representation corresponding to a newform  $f_3$  of weight 2 and odd level. More precisely, the lift is constructed using Theorem 5.1 in loc. cit., and its modularity follows from Theorem 4.1 in loc. cit.

Since  $f_3$  has odd level, we know that it can be base changed to  $F$ . Let us now explain why the chain we have just constructed linking  $f$  to  $f_3$  is safe, in the sense that it allows to propagate modularity for the restrictions to  $G_F$ , backwards. It is enough to concentrate on the part of the chain that goes

from  $f'$  to  $f_3$ , because we have already explained (when dealing with the odd level case) that the part linking  $f$  to  $f'$  is safe (the fact that now the prime 2 is in the level does not affect the argument).

The restrictions to  $G_F$  of the modular Galois representations being considered have residual images that are adequate, even after restriction to  $G_{F(\zeta_p)}$ , when the residual characteristic is  $p = 3$ , and non-solvable and not projectively  $A_5$  when  $p = 2$ . For the mod 3 congruence between  $f'$  and  $f_2$ , we easily see that the A.L.T. in [BLGGT] (its variant in Appendix B) applies over  $F$ . Thus, it remains to check that in the mod 2 congruence modularity over  $F$  propagates from  $f_3$  to  $f_2$ , applying the modification of Theorem 0.9 in [K-2] discussed above. Conditions (1) (residual modularity and non-solvable residual image) and (2) (potentially Barsotti-Tate at primes above 2) of this theorem are satisfied by the restriction to  $G_F$  of the 2-adic representation attached to  $f_2$ , and concerning condition (3'), if we assume that the 2-adic Galois representation attached to  $f_2$  satisfies this potential ordinarity condition, then it is known that this can only happen if it is nearly-ordinary at 2 (cf. [H], Prop. 3.3), but this implies that the residual mod 2 representation is ordinary, thus that the Barsotti-Tate representation attached to  $f_3$  is ordinary.

We conclude that the 2-adic M.L.T. of Kisin applies, thus that modularity propagates well over  $F$ , from  $f_3$  to  $f$ .

Let us now treat case (i). This time we move directly to characteristic 2 and we reduce modulo 2. If the residual Serre's weight is 2, we take a lift corresponding to an odd level, weight 2 modular form  $f_2$  (the existence of which follows from Theorems 4.1 and 5.1 of [KW]), and we see as in the previous paragraph that the 2-adic M.L.T. of Kisin allows to propagate modularity over  $F$  from  $f_2$  to  $f'$ . If the residual Serre's weight is 4, we take a lift, whose existence is also guaranteed by Theorems 4.1 and 5.1 of loc. cit., corresponding to a newform  $f_2$  of weight 2 and even level whose level is strictly divisible by 2 and the prime 2 is Steinberg. Since  $f_2$  falls in case (ii), we have already shown that it can be base changed to  $F$ . It remains to check that the 2-adic M.L.T. of Kisin can be applied to propagate modularity over  $F$  from  $f_2$  to  $f'$ , and again condition (3') is the only non-trivial one. But since  $f_2$  is semistable at 2, it is also ordinary at 2, so condition (3') holds automatically.

We conclude that, in any case, we can construct a safe chain linking  $f$  with an odd level newform, and in particular, that  $f$  can also be base changed to  $F$ .

## 6 Base change for $\mathrm{Symm}^5(\mathrm{GL}(2))$

For level 1 newforms  $f$  we have shown automorphy of  $\mathrm{Symm}^5(f)$ , let us see that by combining this with the base change result in the previous section we can also deduce that  $\mathrm{Symm}^5(f)$  can be base-changed to any totally real number field  $F$ .

This follows by considering the same chain constructed in section 3, and the proof of automorphy for the base case given in section 4, and checking that all the construction can be base changed to  $F$ . The fact that the safe chain also works well over  $F$  is automatic, if we argue as in the previous section: the local conditions for the A.L.T are preserved by base change, and after restriction to  $G_F$  a residual image that is 6-extra large remains 6-extra large. Concerning the base case, the projective residual image  $A_5$  is not changed by restriction to  $G_F$  (again, you can use the fact that the projective image of the restriction to  $G_{\hat{F}}$  is not trivial because the representation is odd and the Galois closure  $\hat{F}$  of  $F$  is totally real, therefore since  $A_5$  is simple the projective image of the restriction to  $G_{\hat{F}}$  does not change, thus clearly for the restriction to  $G_F$  the same holds). The rest of the proof of automorphy for the base case, this time over  $F$ , goes exactly as in section 4, just notice that  $f_0$  and  $f_1$  can be base changed to  $F$  because of the result in the previous section, and that the known cases of Langlands functoriality applied to conclude that  $\mathrm{Symm}^2(f_1) \otimes f_0$  is automorphic are also known to hold when  $f_0$  and  $f_1$  are Hilbert newforms over  $F$ .

## 7 (Conditional) Automorphy for $\mathrm{Symm}^5(\mathrm{GL}(2))$ for newforms of level prime to 30

Using the method of killing ramification as described in section 5, it should be possible to extend the proof of cuspidal automorphy of  $\mathrm{Symm}^5(\mathrm{GL}(2))$  for any non-CM newform whose level is prime to 30. The same applies to the base-changed version of this result discussed in the previous section. The problem is that at some steps (at the true “killing ramification”) we have applied the results in [K] and there is no analogue of such results for representations of dimension greater than 2. At those congruences where the results of [K] have been applied, one of the representations in the congruence is potentially crystalline, but it is not known whether or not it is potentially

diagonalizable. More precisely (cf. Proposition 4.3 in [Di12b]), after the process of ramification swapping we must perform the killing ramification process for a potentially crystalline representation of non-trivial weight  $k$  in residual characteristic  $p$  such that the representation becomes crystalline over a subextension of  $\mathbb{Q}_p(\zeta_p)$  of degree  $e$  and it holds:

$$(k - 1) \cdot e < p - 1$$

Thus, IF WE SUPPOSE that all potentially crystalline representations satisfying this condition are potentially diagonalizable, we can apply the A.L.T. in [BLGGT] and in Appendix B to their 5-th symmetric powers and, combining killing ramification with the safe chain constructed in section 3 (see section 5 for details on such a concatenation) the proof of automorphy for the 5-th symmetric power extends to any non-CM newform of level prime to 30, and arbitrary weight. We are taking level prime to 30 because we need to avoid working in small characteristics ( $p \leq 5$ ) at all steps of the proof.

Of course, another option is that new A.L.T. are proved that can be applied in situations other than the potentially diagonalizable or ordinary cases, such that they apply to the 5-th symmetric power of any potentially crystalline representation satisfying the above condition.

To simplify matters even more, it is easy to see that by a few more applications of “ramification swapping” one can reduce the application of the results in [K] to the case of potentially crystalline representations satisfying the above condition but with the further restriction  $e = 2$ . We do not give here the details of this extra swapping because it is hard to predict whether or not this extra improvement will turn out to be useful.

## 8 Bibliography

- [BLGGT] Barnet-Lamb, T., Gee, T., Geraghty, D., Taylor, R., *Potential automorphy and change of weight*, preprint; available at [www.arxiv.org](http://www.arxiv.org)
- [CHT] Clozel, L., Harris, M., Taylor, R., *Automorphy for some  $\ell$ -adic lifts of automorphic mod  $\ell$  representations*, Pub. Math. IHES **108** (2008), 1-181.
- [Di09] Dieulefait, L., *The level 1 case of Serre’s conjecture revisited*, Rendiconti Lincei - Mat. e Appl. **20** (2009), 339-346
- [Di12a] Dieulefait, L., *Langlands base change for  $GL(2)$* , Annals of Math. **176** (2012) 1015-1038
- [Di12b] Dieulefait, L., *Remarks on Serre’s modularity conjecture*, Manuscripta

- Math. **139** (2012) 71-89
- [DG] Dieulefait, L., Gee, T., *Automorphy lifting for small  $l$* , Appendix B to this paper
- [GL] Gao, H., Liu, T., *A note on potential diagonalizability of crystalline representations*, preprint
- [GK] Gee, T., Kisin, M., *The Breuil-Mezard conjecture for potentially Barsotti-Tate representations*, preprint
- [GJ] Gelbart, S., Jacquet, H., *A relation between automorphic representations of  $GL(2)$  and  $GL(3)$* , Ann. Scient. Ecole. Norm. Sup. **11** (1979) 471-542
- [G] Guralnick, R., *Adequacy of representations of finite groups of Lie type*, Appendix A to this paper
- [GHTT] Guralnick, R., Herzig, F., Taylor, R., Thorne, J., *Adequate subgroups*, J. Inst. Math. Jussieu, arXiv:1107.5993 (to appear), appendix to “On the automorphy of  $\ell$ -adic Galois representations with small residual image” by J.Thorne, J. Inst. Math. Jussieu, arXiv:1107.5989 (to appear)
- [H] Hida, H., *A finiteness property of abelian varieties with potentially ordinary good reduction*, J. Amer. Math. Soc. **25** (2012) 813-826
- [Kim] Kim, H., *An example of non-normal quintic automorphic induction and modularity of symmetric powers of cusp forms of icosahedral type*, Invent. Math. **156** (2004), 495-502
- [KS] Kim, H., Shahidi, F., *Functorial products for  $GL(2) \times GL(3)$  and the symmetric cube for  $GL(2)$* , Annals of Math. **155** (2002) 837-893
- [KiW] Kiming, I., Wang, X., *Examples of 2-dimensional odd Galois representations of  $A_5$ -type over  $\mathbb{Q}$  satisfying the Artin conjecture*, in “On Artin’s Conjecture for odd 2-dimensional representations”, G. Frey (Ed.), LNM 1585, (1994) Springer-Verlag
- [K-BT] Kisin, M., *Moduli of finite flat group schemes, and modularity*, Annals of Math. **170** (2009) 1085-1180
- [K] Kisin, M., *The Fontaine-Mazur conjecture for  $GL_2$* , J.A.M.S **22** (2009) 641-690
- [K-2] Kisin, M., *Modularity of 2-adic Barsotti-Tate representations*, Invent. Math. **178** (2009) 587-634
- [Kh] Khare, C., *Serre’s modularity conjecture: The level one case*, Duke Math. J. **134** (2006) 557-589
- [KW] Khare, C., Wintenberger, J-P., *Serre’s modularity conjecture (I)*, Invent. Math. **178** (2009) 485-504
- [Ri85] Ribet, K., *On  $\ell$ -adic representations attached to modular forms. II*,

Glasgow Math. J. **27** (1985) 185-194

[Ri97] Ribet, K., *Images of semistable Galois representations*, Pacific J. Math. **181** (1997)

[W] Wang, S. *On the symmetric powers of cusp forms on  $GL(2)$  of icosahedral type*, Int. Math. Res. Not. (2003), 2373-2390.